Übungen zur Indextheorie

Universität Regensburg, Wintersemester 2016/17
Prof. Dr. Bernd Ammann/ PD Dr.habil. Olaf Müller
Exercise Sheet 3, due to 10.11.2016

Exercise 1

Let M be an oriented n-dimensional Riemannian manifold, let W be a Clifford bundle over M.

1. We define $\mathbf{Vol} \in Cl(T_pM)$ by $e_1 \cdot \ldots \cdot e_n$ for a positively oriented orthonormal base. Show that this definition doees not depend on which base we choose.
2. Show that, for every smooth section $\phi \in \Gamma(W)$ we have

$$D(\mathbf{Vol} \cdot \phi) = (-1)^{n-1}\ \mathbf{Vol} \cdot (D\phi).$$

Exercise 2

Let (M, g) be a Riemannian manifold with Clifford bundle W. Show that for every $\phi \in \Gamma(W)$:

$$|D\phi|^2 \leq n|\nabla\phi|^2$$

pointwise on M.
Hint: Use the Cauchy-Schwarz inequality to show

$$\Big(\sum_i |\nabla_{e_i}\phi|\Big)^2 \leq n \cdot \sum_i |\nabla_{e_i}\phi|^2.$$

Exercise 3

Let $\phi : T^n \to T^n$ be a diffeomorphism of the standard torus $T^n = \mathbb{R}^n/(2\pi\mathbb{Z})^n$. Show that for all $k \in \mathbb{N}$:

1. $C^k(T^n) \to C^k(T^n), \quad u \mapsto u \circ \phi$ is well-defined and continuous.
2. $H_k \to H_k, \quad u \mapsto u \circ \phi$ is well-defined and continuous.
   (Hint: Instead of working with $\|\cdot\|_k$ work with the equivalent norm $\sum_{j=0}^{k} \left\| \frac{\partial^{|\alpha|}}{\partial x^\alpha} u \right\|_0$.)

Exercise 4

The following construction of 'mollifiers' will be very useful for us in regularity theory.

1. Choose $f \in C_0^\infty(\mathbb{R}^n, (0,\infty))$ radially symmetric and normed by $\int_{\mathbb{R}^n} f = 1$. For $a > 0$ ddefine $f_a \in C_0^\infty(\mathbb{R}^n, (0,\infty))$ by $f_a(x) := a^{-n} f(x/a)$ for all $x \in \mathbb{R}^n$, and $F_a : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by $F_a s(x) := (f_a * s)(x) = \frac{1}{a^n}\int_{\mathbb{R}^n} f(\frac{x-y}{a}) s(y) dy$. Show that there is $C > 0$ with $||F_a|| < C$ for all $a \in (0,\infty)$.
2. Show that if $s \in C_0^0(\mathbb{R}^n)$ then $\lim_{a\to 0} F_a s = s$ in $C^0(\mathbb{R}^n)$.
3. Conclude that if $s \in L^2(\mathbb{R}^n)$ then $\lim_{a\to 0} F_a s = s$ in $L^2(\mathbb{R}^n)$.
4. Let $b \in C_c^1(\mathbb{R}^n) := \{f \in C^1(\mathbb{R}^n) | \text{supp}(f) \text{ compact}\}$. For $B := b(x)\frac{\partial}{\partial x_1}$, show

$$\begin{aligned}([B, F_a]s)(x) &= a^{-n} \int f(\frac{x-y}{a}) \partial_1 b(y) s(y) dy \\ &+ a^{-(n-1)} \int (b(x) - b(y)) \partial_1 f(\frac{x-y}{a}) s(y) dy.\end{aligned}$$

   Conclude that there is a $C > 0$ such that $||[B, F_a]|| < C$ for all $a \in (0,\infty)$.