Übungen zur Indextheorie

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Exercise 1

Let (V, g) be a complex Hermitean vector space fo finite dimension. Let

$$S := \Lambda^{\bullet}_{\mathbb{C}} V = \bigoplus_{k=0}^{\dim_{\mathbb{C}} V} \Lambda^k_{\mathbb{C}} V$$

be the exterior product in the sense of complex vector spaces. We define for $v \in V$ in analogy to the real case $v^b := g(\,\cdot\,, v)$. Thus v^b is an element of the dual space V^*, but $v \mapsto v^b$ is semilinear, i.e. $\mathbb{R}$-linear and $(iv)^b = -iv^b$. For $\alpha \in \Lambda^k_{\mathbb{C}} V$ we define $v^b \lrcorner \alpha \in \Lambda^{k-1}_{\mathbb{C}} V$ by plugging v^b in the first argument of α viewed as multilinear map $\alpha : V^* \times \cdots \times V^* \to \mathbb{C}$. Further $v \llcorner \alpha := v^b \lrcorner \alpha$. Show that $\Lambda^* V$ is a Clifford module for the Euclidean space $(V, \Re g)$ with the Clifford multiplication $\cdot$ defined by

$$v \cdot \alpha := v \wedge \alpha - v \llcorner \alpha.$$

Exercise 2

Let W be a Clifford module for the Euclidean $\mathbb{R}^n$ and the standard basis $e_1, ..., e_n$. We define $\mathbf{vol} \in \mathbf{End}(W)$ by $\mathbf{vol}(w) := e_1 \cdot ... \cdot e_n \cdot w$ for all $w \in W$.

- Compute $\mathbf{vol}^2$.
- Does $e_k \cdot$ commute or anticommute with $\mathbf{vol}$?
- For $n \in 2\mathbb{N}$, show that there is $v_n \in \{1, i\}$ such that W is a Clifford module for $\mathbb{R}^{n+1}$ with the Clifford multiplication $\tilde{\cdot} : \mathbb{R}^{n+1} \otimes W \to W$ defined by

$$e_k \tilde{\cdot} w = \begin{cases} e_k \cdot w & \forall k \in \{1, ..., n\} \\ v_n \cdot \mathbf{vol}(w) & \text{for } k = n+1. \end{cases}$$

- For $n \in 2\mathbb{N} + 1$, show that $W' = W \oplus W$ is a Clifford module for $\mathbb{R}^{n+1}$ with the Clifford multiplication $\tilde{\cdot} : \mathbb{R}^{n+1} \otimes W' \to W'$ defined by

$$e_k \tilde{\cdot} (w_1, w_2) = \begin{cases} (e_k \cdot w_1, -e_k \cdot w_2) & \forall k \in \{1, ..., n\} \\ (-w_2, w_1) & \text{for } k = n+1. \end{cases}$$

Exercise 3

Let W be a Clifford bundle over a Riemannian manifold (M, g) with Clifford multiplication $\mathrm{cl} : T^*M \otimes W \to W$ (a differential operator of order 0). Compute $\mathrm{cl}^\sharp$ and $\mathrm{cl}^\sharp \circ \mathrm{cl}$.

Exercise 4

1. Assume that there is a constant $C > 0$ such that for all 1-forms ω we have

$$\langle \mathcal{K}^1 \omega, \omega \rangle \geq C \langle \omega, \omega \rangle,$$

   where $\mathcal{K}^1$ is the curvature endomorphism. Let $f \in C^\infty(M)$ be an eigenfunction of Δ for an eigenvalue $\lambda \neq 0$. Show that

$$\lambda \geq C.$$

2. Let (M, g) be a Riemannian manifold. The metric g defines a bundle metric on the cotangential bundle $T^*M \to M$ via the product rule. Compute its curvature and the curvature endomorphism $\mathcal{K}_1$ on forms!

3. Conclude a lower estimate for the first eigenvalue of the Laplace operator on smooth real-valued functions.