BOTT PERIODICITY IN K-THEORY OF C^*-ALGEBRAS

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Abstract. The aim of this set of talks is to understand the proof of Bott periodicity due to Cuntz [2].

(1) **Talk 1:** Introduce the category of C^*-algebras. Its objects are C^*-algebras and its morphisms are $*$-homomorphisms. Explain the notion of positive elements and that of (minimal) C^*-tensor product $\hat{\otimes}$. Establish the following:
   - existence of kernel, cokernel, and (countable) inductive limits,
   - $*$-homomorphisms are norm decreasing.

   Remark. You may need to treat as black-boxes a few results like functional calculus, description of positive elements, spectral radius formula for normal elements, etc..

   UPSHOT: In the category of C^*-algebras a sequence $0 \to A \to B \to C \to 0$ is short exact if and only if it is algebraically short exact.

   Reference: Chapter 1 of [4]

(2) **Talk 2:** Introduce K_0-theory of C^*-algebras in two steps - construct the semigroup of projections $P_\infty(A)$ for unital A and then apply the Grothendieck group functor. Explain the extension to nonunital C^*-algebras. Briefly sketch functoriality and then establish the following:
   - half-exactness of K_0-theory,
   - C^*-stability: $\mathrm{K}_0(A\hat{\otimes}\mathbb{K}) \cong \mathrm{K}_0(A)$, where $\mathbb{K} = \varinjlim_n M_n(\mathbb{C})$.

   If time permits talk about Serre-Swan theorem.

   Remark. The K_0-group can be constructed more generally for arbitrary rings via idempotents. You can also take this approach. You may assume the continuity of K_0-theory, viz., $\mathrm{K}_0(\varinjlim_n A_n) \cong \varinjlim_n \mathrm{K}_0(A_n)$.

   Reference: The material is scattered across chapters 2, 3 and 4 of [5] (you may also consult [3])

(3) **Talk 3:** Set $\Sigma A = \mathrm{C}_0((0,1))\hat{\otimes}A$. Establish the Puppe sequence for K_0-theory, i.e., for every short exact sequence $0 \to A \to B \to C \to 0$ there is a long exact sequence

$$\cdots \to \mathrm{K}_0(\Sigma A) \to \mathrm{K}_0(\Sigma B) \to \mathrm{K}_0(\Sigma C) \to \mathrm{K}_0(A) \to \mathrm{K}_0(B) \to \mathrm{K}_0(C)$$

   For this briefly sketch the homotopy invariance of K_0-theory. Combined with half exactness of K_0-theory one obtains the desired Puppe sequence (see 21.4 of [1])

   Introduce the Toeplitz extension $0 \to \mathbb{K} \to \mathfrak{T} \to \mathrm{C}(S^1) \to 0$. Talk a bit about the theory of universal C^*-algebras; the Toeplitz algebra $\mathfrak{T}$ is the universal unital C^*-algebra generated by one isometry [Coburn].

   Reference: Chapter 4 of [3] and chapters 3-4 of [5]

(4) **Talk 4:** Set inductively $\Sigma^n A = \Sigma(\Sigma^{n-1} A)$. The Toeplitz extension $0 \to \mathbb{K} \to \mathfrak{T} \to \mathrm{C}(S^1) \to 0$ gives rise to a reduced Toeplitz extension $0 \to \mathbb{K} \to \mathfrak{T}_0 \to \mathrm{C}_0((0,1)) \to 0$. Now do the following to complete the proof of Bott periodicity:

- show that $\mathrm{K}_0(\Sigma^n A) = 0$,
- apply $-\hat{\otimes} A$ to the reduced Toeplitz extension and insert the value of $\mathrm{K}_0(\Sigma^n A)$ in the Puppe sequence associated with it, and
- use C^*-stability of K_0-theory.

Remark. The technique is very general and actually shows that any (abelian group valued) functor F on the category of C^*-algebras that satisfies homotopy invariance, C^*-stability, and half-exactness is Bott periodic, i.e., for any C^*-algebra A one has $F(\Sigma^2 A) \cong F(A)$.

Reference: [2] or Chapter 4 of [3]

References

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