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WS 2012/13
15.10.2012

Differentialgeometrie I
Anwesenheitsaufgaben am 17. und am 18.10.2012

Aufgabe 1

Bilden Sie für $A \subset \mathbb{R}$ das Innere und den Abschluss, und zwar bzgl. der Standardtopologie, der diskreten Topologie und der Klumpentopologie auf $\mathbb{R}$, wobei

1. $A = [0, 1]$,
2. $A = (0, 1)$,
3. $A = \mathbb{Z}$,
4. $A = \mathbb{Q}$.

Aufgabe 2

Es tragen $\mathbb{R}^n$ und $\mathbb{R}^m$ die Standardtopologien. Zeigen Sie, dass die Produkttopologie auf $\mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}$ mit der Standardtopologie übereinstimmt.

Aufgabe 3

Sei $(X, \mathcal{O})$ ein topologischer Raum, "$\sim$" eine Äquivalenzrelation auf X, $X/\sim = \{[x] \mid x \in X\}$ sei die Menge der Äquivalenzklassen, $\pi : X \to X/\sim$, $\pi(x) = [x]$.

1. Zeigen Sie, dass die Quotiententopologie $\mathcal{O}'$ auf $X/\sim$ tatsächlich eine Topologie ist.
2. Zeigen Sie: $\pi : X \to X/\sim$ ist stetig.
3. Sei $(Y, \mathcal{O}_Y)$ ein weiterer topologischer Raum und $f : X/\sim \to Y$ eine Abbildung. Zeigen Sie:
$$f : X/\sim \to Y \text{ ist stetig} \quad \Longleftrightarrow \quad f \circ \pi : X \to Y \text{ ist stetig.}$$
4. Sei $X = [0, 1] \subset \mathbb{R}$ mit der Standardtopologie. Wir definieren
$$s \sim t \quad \Longleftrightarrow \quad \begin{cases} s = t \text{ oder} \\ s = 0 \text{ und } t = 1, \text{ oder} \\ s = 1 \text{ und } t = 0. \end{cases}$$
Zeigen Sie: $[0, 1]/\sim$ mit der Quotiententopologie ist homöomorph zu $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ mit der Standardtopologie.

Aufgabe 4
Sei $(X, \mathcal{O}_X)$ ein Hausdorff-Raum, sei p ein weiterer Punkt, $p \notin X$. Setze $X^+ := X \cup \{p\}$. Wir definieren

$$\mathcal{O}_{X^+} := \mathcal{O}_X \cup \left\{ (X \setminus K) \cup \{p\} \;\middle|\; K \subset X \text{ kompakt} \right\} .$$

Zeigen Sie:

1. $(X^+, \mathcal{O}_{X^+})$ ist ein topologischer Raum.
2. X^+ ist kompakt.
3. X^+ ist Hausdorffsch genau dann, wenn X *lokalkompakt* ist, d. h. jeder Punkt in x besitzt eine kompakte Umgebung.
4. $(-1, 1)^+$ ist homöomorph zu S^1.

Man nennt X^+ die *Ein-Punkt-Kompaktifizierung* von X.

Aufgabe 5
Sei $f : X \longrightarrow Y$ eine stetige Abbildung zwischen metrischen Räumen.

1. Ist das Urbild jeder kompakten Teilmenge von Y eine kompakte Teilmenge von X? Begründen Sie Ihre Antwort.
2. Im Falle $X = \mathbb{R}^n$ und $Y = \mathbb{R}^m$ nehmen wir zusätzlich voraus, dass $|f(x)| \underset{|x| \to \infty}{\longrightarrow} \infty$, wobei $|\cdot|$ die euklidische Norm im $\mathbb{R}^n$ bezeichnet. Zeigen Sie, dass dann $f^{-1}(K)$ kompakt ist für jedes Kompaktum $K \subset \mathbb{R}^m$.

Aufgabe 6
Sei $f : X \longrightarrow Y$ eine stetige Abbildung zwischen topologischen Räumen.

1. Zeigen Sie, dass das Bild einer zusammenhängenden Teilmenge von X unter f wieder zusammenhängend ist.
2. Ist das Urbild jeder zusammenhängenden Teilmenge von Y unter f wieder zusammenhängend? Begründen Sie Ihre Antwort.

Aufgabe 7
Die Sinuskurve der Topologen ist ein topologischer Raum, definiert als der Graph der Funktion $\sin\left(\frac{1}{x}\right)$ im halboffenen Intervall $(0, 1]$, zusammen mit $\{0\} \times [-1, 1]$, mit der Standard-Unterraumtopologie:

$$T = \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) : x \in (0, 1] \right\} \bigcup \{(0, y) : y \in [-1, 1]\}.$$

Zeigen Sie:

1. T ist zusammenhängend.
2. T ist weder lokal zusammenhängend noch wegzusammenhängend.

Aufgabe 8

Sei Y eine mit der Relativtopologie versehene nichtleere Teilmenge eines topologischen Raumes X und $f : Y \longrightarrow \mathbb{R}$ eine stetige Abbildung.

1. Gibt es immer eine stetige Abbildung $\widetilde{f} : X \longrightarrow \mathbb{R}$ so, dass $\widetilde{f}_{|Y} = f$ gilt? Begründen Sie Ihre Antwort.

2. Zeigen Sie, dass im Fall $Y = S^n := \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$ und $X = \mathbb{R}^{n+1}$ jede stetige Abbildung $f : Y \longrightarrow \mathbb{R}$ sich zu einer stetigen Abbildung $\widetilde{f} : X \longrightarrow \mathbb{R}$ fortsetzen lässt.

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Differentialgeometrie I
1. Übungsblatt

Aufgabe 1

(a) Sei M eine glatte n-dimensionale Untermannigfaltigkeit von $\mathbb{R}^m$. (Wir benutzen „glatt" immer im Sinne von C^∞.) Wenden Sie den Satz über implizite Funktionen an, um zu zeigen, dass M eine n-dimensionale C^∞-Mannigfaltigkeit ist.

(b) Sei nun $M := S^n := \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ die n-Sphäre. Geben Sie für jeden Punkt $p \in S^n$ eine offene Umgebung V von p in S^n an und einen Homöomorphismus von V nach $\mathbb{R}^n$.

Aufgabe 2

Sei $n \in \mathbb{N}$. Es bezeichne $\mathbb{R}\mathrm{P}^n$ die Menge der 1-dimensionalen Untervektorräume von $\mathbb{R}^{n+1}$.

(a) Man identifiziere $\mathbb{R}\mathrm{P}^n$ mit der Quotientenmenge $\mathbb{R}^{n+1} \setminus \{0\}/_\sim$, wobei $x \sim y \iff \exists\, \lambda \in \mathbb{R}^\times$ s.d. $x = \lambda y$, und versehen es mit der Quotiententopologie. Zeigen Sie, dass $\mathbb{R}\mathrm{P}^n$ ein kompakter Hausdorff-Raum ist, der das 2. Abzählbarkeitsaxiom erfüllt.

(b) Zeigen Sie, dass die Abbildungen

$$U_j := \{[x] \in \mathbb{R}\mathrm{P}^n \mid x_j \neq 0\} \xrightarrow{\varphi_j} \mathbb{R}^n, \quad [x] \longmapsto \frac{1}{x_j}(x_1, \ldots, \widehat{x_j}, \ldots, x_{n+1})$$

(mit $1 \leq j \leq n+1$), wohldefinierte Homöomorphismen sind (mit "$\widehat{x_j}$" ist "ohne x_j" gemeint).

(c) Zeigen Sie, dass dieser Atlas C^ω ist und leiten Sie daraus her, dass $\mathbb{R}\mathrm{P}^n$ eine Struktur einer n-dimensionalen C^ω-Mannigfaltigkeit trägt.

Aufgabe 3

Sei $k \in \mathbb{N} \cup \{0, \infty, \omega\}$. Zeigen Sie, dass jeder C^k-Atlas $\mathcal{A}$ in genau einer C^k-Struktur $\overline{\mathcal{A}}$ enthalten ist. Angenommen $\mathcal{A}_1$ und $\mathcal{A}_2$ sind zwei C^k-Atlanten von M. Zeigen Sie: $\overline{\mathcal{A}_1} = \overline{\mathcal{A}_2}$ genau dann, wenn alle Karten in $\mathcal{A}_1$ mit allen Karten in $\mathcal{A}_2$ C^k-verträglich sind.

Aufgabe 4
Ein topologischer Raum X heißt genau dann *wegzusammenhängend*, wenn je zwei Punkte aus X durch einen stetigen Weg in X verbunden werden können. Ein topologischer Raum X heißt genau dann *lokal wegzusammenhängend*, wenn jeder Punkt aus X eine Basis von wegzusammenhängenden Umgebungen besitzt.

1. Zeigen Sie, dass jede topologische Mannigfaltigkeit lokal wegzusammenhängend ist.

2. Zeigen Sie, dass die Zusammenhangskomponenten eines lokal wegzusammenhängenden topologischen Raumes offen und abgeschlossen sind.

3. Leiten Sie daraus her, dass die Zusammenhangskomponenten einer m-dimensionalen topologischen Mannigfaltigkeit selbst m-dimensionale topologische Mannigfaltigkeiten sind.

Abgabe der Lösungen: ***Montag, den 22.10.2012*** *in der Vorlesung.*

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WS 2012/13
22.10.2012

Differentialgeometrie I
2. Übungsblatt

Aufgabe 1
Definieren Sie eine C^k-Struktur auf dem Produkt $M \times N$ zweier C^k-Mannigfaltigkeiten M und N so, dass die Projektionen $M \times N \longrightarrow M$ und $M \times N \longrightarrow N$ C^k-Abbildungen sind.

Aufgabe 2
Zeigen Sie, dass ein topologischer Raum X genau dann lokal homöomorph zu $\mathbb{R}^n$ ist, wenn jeder Punkt von X eine zu $\mathbb{R}^n$ selbst homöomorphe offene Umgebung besitzt.

Aufgabe 3
Wir betrachten $\mathbb{Z}^n$ als Untergruppe von $\mathbb{R}^n$. Wie in der Gruppentheorie definieren wir die Relation $x \sim y \iff x - y \in \mathbb{Z}^n$ für $x, y \in \mathbb{R}^n$. Der dazu gehörige Quotient $M^n := \mathbb{R}^n/\mathbb{Z}^n$ heißt *n-dimensionaler Torus.*

(a) Zeigen Sie, dass M^n ein kompakter Hausdorff-Raum ist.

(b) Konstruieren Sie einen C^∞-Atlas auf M^n, so dass die kanonische Projektion $\pi : \mathbb{R}^n \longrightarrow M^n$ glatt und lokal glatt umkehrbar ist.

(c) Zeigen Sie, dass M^n diffeomorph zu $\underbrace{S^1 \times \ldots \times S^1}_{n\text{-mal}}$ ist.

(d) Zeigen Sie im Fall $n = 2$, dass M^2 diffeomorph zum Drehtorus $\{(x, y, z)^T \in \mathbb{R}^3 \mid (\sqrt{x^2 + y^2} - 2)^2 + z^2 = 1\}$ ist.

Aufgabe 4
Es bezeichnen $\mathbb{R}^{n\times n}$ den Raum aller reellen $n \times n$-Matrizen und $\mathbb{R}^{n\times n}_{\text{sym}}$ den Unterraum aller symmetrischen $n \times n$-Matrizen.

1. Sei $f : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n}_{\text{sym}}$, $A \longmapsto {}^tAA$, wobei ${}^tA$ die zu A transponierte Matrix bezeichnet. Zeigen Sie, dass die Identitätsmatrix $\mathbb{1}_n$ ein regulärer Wert von f ist.
   Zur Erinnerung: $c \in \mathbb{R}^{n\times n}_{\text{sym}}$ *ist genau dann ein regulärer Wert, wenn für jedes* $x \in f^{-1}(\{c\})$ *das Differential* $d_x f$ *vollen Rang hat.*

2. Bestimmen Sie $\mathrm{Ker}(d_{\mathbb{1}_n} f)$.

3. Folgern Sie, dass die orthogonale Gruppe O_n eine $\frac{n(n-1)}{2}$-dimensionale Untermannigfaltigkeit von $\mathbb{R}^{n\times n} \cong \mathbb{R}^{n^2}$ ist.

4. Bestimmen Sie eine Karte von O_n, die $\mathbb{1}_n$ enthält.
   Hinweis: betrachten Sie die Exponentialabbildung.

Abgabe der Lösungen: ***Montag, den 29.10.2012*** *in der Vorlesung.*

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Differential Geometry I
Exercise Sheet no. 3

Exercise 1
Let M^m be an m-dimensional submanifold of $\mathbb{R}^k$ an $p \in M^m$ be a point. Prove that the tangent space of the manifold M^m at p as defined in the lecture can be identified with the tangent space of the submanifold M^m at p you already know from previous lectures.

Exercise 2
Show that any topological manifold carries an atlas with countably many charts.

Exercise 3
Let M be a set and $n \in \mathbb{N}$. Further, we assume that a family of bijective maps $(\phi_\alpha : U_\alpha \to V_\alpha)_{\alpha \in A}$ is given, where U_α is a subset of M and where V_α is an open subset of $\mathbb{R}^n$. This family is supposed to satisfy:

(i) $M = \bigcup_{\alpha \in A} U_\alpha$,

(ii) $\phi_\alpha(U_\alpha \cap U_\beta)$ is open in $\mathbb{R}^n$ for all $\alpha, \beta \in \mathbb{R}^n$

(iii) $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is continuous for all $\alpha, \beta \in A$.

Show that

(a) There is a unique topology on M such that all U_α are open and such that all ϕ_α are homeomorphisms.

(b) If $A_1 \subset A$ satisfies $M = \bigcup_{\alpha \in A_1} U_\alpha$, then $(\phi_\alpha : U_\alpha \to V_\alpha)_{\alpha \in A}$ and $(\phi_\alpha : U_\alpha \to V_\alpha)_{\alpha \in A_1}$ induce the same topology on M.

(c) The topology on M is second countable if A is countable.

(d) Suppose that for any $p, q \in M$ we have:

   (i) there is $\alpha \in A$ with $p, q \in U_\alpha$, **or**

   (ii) there are $\alpha, \beta \in A$ with $p \in U_\alpha$, $q \in U_\beta$, $U_\alpha \cap U_\beta = \emptyset$.

   Then the topology on M is Hausdorff.

Are the sufficient conditions in (c) resp. (d) for second countability resp. Hausdorff property also necessary?

Exercise 4
Let $\mathbb{C}\mathrm{P}^n := {}^{\textstyle\mathbb{C}^{n+1} \smallsetminus \{0\}}\!/\!_{\sim}$ denote the complex projective space where, by definition, $x \sim y \iff x \in \mathbb{C} \cdot y$. We note by $[x^1, \dots, x^{n+1}]$ the equivalence class of $(x^1, \dots, x^{n+1}) \in \mathbb{C}^{n+1} \smallsetminus \{0\}$. For $\alpha \in \{1, \dots, n+1\}$ we let U_α be the subset of all $[x^1, \dots, x^{n+1}] \in \mathbb{C}\mathrm{P}^n$ with $x^\alpha \neq 0$ and define the map

$$\begin{aligned} \phi_\alpha : U_\alpha &\longrightarrow \mathbb{C}^n \\ [x^1, \dots, x^{n+1}] &\longmapsto \left(\frac{x_1}{x_\alpha}, \dots, \widehat{x_\alpha}, \dots, \frac{x_{n+1}}{x_\alpha} \right), \end{aligned}$$

where, as usual, $\widehat{x_\alpha}$ means that the α^{th} coordinate is omitted.

1. Show that U_α and ϕ_α are well-defined and that ϕ_α is bijective, for all $\alpha \in \{1, \dots, n+1\}$.

2. Show with the help of Exercise 3 that $\mathcal{A} := \{(U_\alpha, \phi_\alpha), 1 \leq \alpha \leq n+1\}$ defines a structure of C^∞ manifold on $\mathbb{C}\mathrm{P}^n$.

3. Show that the underlying topology coincides with the quotient topology, where $\mathbb{C}^{n+1}$ carries the standard topology.

Abgabe der Lösungen: ***Montag, den 5.11.2012*** *vor der Vorlesung.*

Universität Regensburg, Mathematik WS 2012/13
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Differential Geometry I
Exercise Sheet no. 4

Exercise 1
Let X be a tangent vector field on a smooth manifold M, that is, X is a map $M \to TM$ with $\pi \circ X = \mathrm{Id}_M$, where $\pi : TM \to M$ is the projection map. Recall that, given a chart $\varphi : U \to V$ of M, the associated coordinate vector fields $\{\frac{\partial}{\partial\varphi^1}, \ldots, \frac{\partial}{\partial\varphi^n}\}$ form a basis of TM in each point of U, in particular any tangent vector field X on M can be written in the form $X = X^i \frac{\partial}{\partial\varphi^i}$ on U, where $X^1, \ldots, X^n : U \to \mathbb{R}$ are functions.
Show that X is smooth as a map between manifolds if and only if the functions $X^1, \ldots, X^n : U \to \mathbb{R}$ are smooth for any chart.

Exercise 2
Let $f : M \to \mathbb{R}$ be a C^1 map on a compact smooth n-dimensional manifold, where $n \geq 1$.

1. Let $p \in M$ be a point. Show that, if f reaches a maximum or a minimum at p, then $d_p f = 0$.

2. Show that the differential map of f vanishes in at least two points in M.

3. Show that f has exactly one critical value if and only if f is constant.

Exercise 3
Let M be a compact smooth n-dimensional manifold. By definition, a *one-parameter group of diffeomorphisms* on M is a smooth map $\varphi : M \times \mathbb{R} \to M$, $(x, t) \mapsto \varphi_t(x)$, with $\varphi_0 = \mathrm{Id}_M$ and $\varphi_t \circ \varphi_s = \varphi_{t+s}$ for all $s, t \in \mathbb{R}$.

1. Show that, given any one-parameter group of diffeomorphisms $(\varphi_t)_t$ on M, the map $X(x) := \frac{d}{dt}|_{t=0}(\varphi_t(x))$ defines a smooth tangent vector field on M.

2. Conversely, show that, given any smooth vector field X on M, there exists a unique one-parameter group of diffeomorphisms $(\varphi_t)_t$ on M such that $\frac{d}{dt}|_{t=0}(\varphi_t(x)) = X(x)$ for all $x \in M$.
Hint: First construct $\varphi_t(x)$ for fixed x and t close to 0 using the theorem of Picard-Lindelöff; then show that $t \mapsto \varphi_t(x)$ can be extended on $\mathbb{R}$.

Abgabe der Lösungen: ***Montag, den 12.11.2012*** *vor der Vorlesung.*

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Differential Geometry I
Exercise Sheet no. 5

Exercise 1

Let M be a smooth n-dimensional manifold and, for each point $p \in M$, $g_{|_p}$ be a Euclidean inner product on T_pM. Show that the following statements are equivalent:

1. For any smooth tangent vector fields X, Y on M, the map $M \to \mathbb{R}$, $p \mapsto g_{|_p}(X(p), Y(p))$, is smooth.
2. For any chart $\varphi : U_\varphi \to V_\varphi$ of M and all $1 \leq i, j \leq n$, the function $g^\varphi_{ij} : V_\varphi \to \mathbb{R}$ defined in the lecture is smooth.

Exercise 2

Let M^m, N^n be smooth manifolds and $\phi : M \to N$ be an immersion, that is, ϕ is a smooth map and $d\phi_{|_p} : T_pM \to T_{\phi(p)}N$ is injective for any $p \in M$. Show that, for any Riemannian metric h on N, the map $p \mapsto (d\phi_{|_p})^* h_{|_p}$ introduced in the lecture defines a Riemannian metric on M.

Exercise 3

Let M be a smooth n-dimensional manifold. Recall that a *derivation* on M is a linear map $\delta : C^\infty(M) \to C^\infty(M)$ which satisfies the product rule: for all $f_1, f_2 \in C^\infty(M)$,

$$\delta(f_1 f_2) = (\delta f_1) f_2 + f_1 (\delta f_2).$$

Let X, Y are two smooth tangent vector fields on M.

1. Show that $[\partial_X, \partial_Y] := \partial_X \circ \partial_Y - \partial_Y \circ \partial_X$ defines a derivation on M. Here, ∂_X is the derivation associated to X as in the lecture.
2. Deduce that there exists a unique smooth tangent vector field on M, which we denote by $[X, Y]$, such that $\partial_{[X,Y]} = [\partial_X, \partial_Y]$.
3. Show that, for any $f \in C^\infty(M)$, one has $[X, fY] = \partial_X f \cdot Y + f[X, Y]$.
4. Show that, if $\varphi : U_\varphi \to V_\varphi$ is a chart of M, then $[\frac{\partial}{\partial\varphi^i}, \frac{\partial}{\partial\varphi^j}] = 0$ for all $1 \leq i, j \leq n$. Deduce that, if $X_{|_{U_\varphi}} = X^i \frac{\partial}{\partial\varphi^i}$ and $Y_{|_{U_\varphi}} = Y^i \frac{\partial}{\partial\varphi^i}$, then

$$[X, Y]_{|_{U_\varphi}} = \big(\partial_X(Y^i) - \partial_Y(X^i)\big)\frac{\partial}{\partial\varphi^i} = \Big(X^j \frac{\partial Y^i}{\partial\varphi^j} - Y^j \frac{\partial X^i}{\partial\varphi^j}\Big)\frac{\partial}{\partial\varphi^i}.$$

Exercise 4
Let $\langle\!\langle \cdot\,,\cdot \rangle\!\rangle$ denote the following bilinear form on $\mathbb{R}^{n+1}$:

$$\langle\!\langle x, y \rangle\!\rangle := -x_0 y_0 + \sum_{j=1}^{n} x_j y_j$$

for all $x = (x_0, x_1, \ldots, x_n)$ and $y = (y_0, y_1, \ldots, y_n)$ in $\mathbb{R}^{n+1}$.

1. Show that $\langle\!\langle \cdot\,,\cdot \rangle\!\rangle$ defines a non-degenerate symmetric bilinear form of index 1 on $\mathbb{R}^{n+1}$.

2. Let $\mathbb{H}^n := \{x \in \mathbb{R}^{n+1}\,,\, \langle\!\langle x, x \rangle\!\rangle = -1 \text{ and } x_0 > 0\} \subset \mathbb{R}^{n+1}$. Show that $\mathbb{H}^n$ is a smooth n-dimensional submanifold of $\mathbb{R}^{n+1}$.

3. Prove that, for any $p \in \mathbb{H}^n$, the tangent space of $\mathbb{H}^n$ at p can be canonically identified with $E_p := \{X \in \mathbb{R}^{n+1}\,,\, \langle\!\langle X, p \rangle\!\rangle = 0\}$.

4. Show that $\langle\!\langle \cdot\,,\cdot \rangle\!\rangle_{|E_p \times E_p}$ is positive-definite and deduce that $\langle\!\langle \cdot\,,\cdot \rangle\!\rangle$ induces a Riemannian metric on $\mathbb{H}^n$.

Abgabe der Lösungen: ***Montag, den 19.11.2012*** *vor der Vorlesung.*

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19.11.2012

Differential Geometry I
Exercise Sheet no. 6

Exercise 1
Let $F : M \longrightarrow N$ be a smooth map between smooth manifolds M and N. Let X, Y (resp. $\tilde{X}, \tilde{Y}$) be (smooth) vector fields on M (resp. N). We say that X is *F-related* to $\tilde{X}$ iff $dF \circ X = \tilde{X} \circ F$ holds on M.
Show that, if X is F-related to $\tilde{X}$ and Y is F-related to $\tilde{Y}$, then $[X, Y]$ is F-related to $[\tilde{X}, \tilde{Y}]$.

Exercise 2
Let M^n be a smooth n-dimensional submanifold of $\mathbb{R}^k$. Given vector fields X, Y on M, we extend them to vector fields $\overline{X}, \overline{Y}$ in an open neighbourhood of M in $\mathbb{R}^k$ and set $(\nabla_X Y)_{|_p} := \mathrm{pr}_p\left((\partial_{\overline{X}}\overline{Y})_{|_p}\right)$ for every $p \in M$, where $\mathrm{pr}_p : \mathbb{R}^k \to T_pM$ denotes the orthogonal projection on T_pM (identified to a vector subspace of $\mathbb{R}^k$).
Show that ∇ defines a metric and torsion-free connection on M.

Exercise 3
Recall that an *isometry* between two smooth n-dimensional Riemannian manifolds (M, g) and (N, h) is a diffeomorphism $\varphi : M \to N$ which preserves the metric, that is, with $\varphi^* h = g$.

1. Show that, in the case $(M, g) = (N, h)$, the isometries of (M, g) form a group w.r.t. the composition of maps.

2. Let $\mathrm{Aff}(M, g)$ be the set of diffeomorphisms of M preserving the Levi-Civita connection ∇ of (M, g), that is,

$$\mathrm{Aff}(M, g) := \{\varphi : M \to M \text{ diffeo. with } \nabla_{\varphi_* X}\varphi_* Y = \varphi_*(\nabla_X Y)\ \forall X, Y \in \mathfrak{X}(M)\},$$

   where, for any $X \in \mathfrak{X}(M)$, the vector field $\varphi_* X \in \mathfrak{X}(M)$ is defined by $(\varphi_* X)(x) := d_{\varphi^{-1}(x)}\varphi(X(\varphi^{-1}(x)))$ for all $x \in M$. Show that $\mathrm{Aff}(M, g)$ is a group containing the group of isometries of (M, g).

3. Let $M = \mathbb{R}^n$ be endowed with the standard Riemannian metric g, i.e. the Euclidean metric. Determine all elements $\varphi \in \mathrm{Aff}(M, g)$ with $\varphi(0) = 0$.

Exercise 4

Let M be a smooth manifold. Given a 1-parameter group of diffeomorphisms $\varphi : M \times \mathbb{R} \to M$, $(x,t) \mapsto \varphi_t(x)$ on M, let X be associated tangent vector field on M as in Exercise no. 3 of Sheet 4. Show that, for any smooth tangent vector field Y on M,

$$\frac{d}{dt}_{|t=0}(\varphi_t)_* Y = -[X,Y],$$

where, for any $t \in \mathbb{R}$, $(\varphi_t)_* Y$ denotes the push-out tangent vector field of Y defined in the last exercise above.

Abgabe der Lösungen: ***Montag, den 26.11.2012*** *vor der Vorlesung.*

Universität Regensburg, Mathematik WS 2012/13
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Differential Geometry I
Exercise Sheet no. 7

Exercise 1

Let (M, g) be a smooth compact Riemannian manifold. Show that every geodesic of (M, g) is defined in $\mathbb{R}$.

Exercise 2

1. Let M_1 and M_2 be two smooth surfaces in $\mathbb{R}^3$, and assume that we have a smooth curve $c : I \to \mathbb{R}^3$ with $c(I) \subset M_1 \cap M_2$. Further we assume that $T_{c(t)}M_1 = T_{c(t)}M_2$ for all $t \in I$. Show that the parallel transports along c in M_1 and in M_2 coincide.

2. Given $\theta \in]0, 2\pi[$ let $C := \{(r\cos\varphi, r\sin\varphi)\,,\ r \in]0,\infty[,\ \varphi \in]0,\theta[\} \subset \mathbb{R}^2$. Determine the parallel transport along the curve $c_r :]0,\theta[\to C$, $t \mapsto (r\cos t, r\sin t)$, where $r > 0$.

3. Deduce an explicit formula for the parallel transport along a circle of latitude $t \mapsto (\cos t\cos\varphi, \sin t\cos\varphi, \sin\varphi)$ in S^2, where $\varphi \in]-\frac{\pi}{2}, \frac{\pi}{2}[$.

Exercise 3 *(Geodesics in hyperbolic space)*

1. Let $\phi : M \to M$ be an isometry of (M, g). For any $X \in TM$ let $\gamma_X : I \to M$ be a geodesic with $\dot\gamma_X(0) = X$.
   Show: $\phi(\gamma_X(t)) = \gamma_X(t)$ for all $t \in I$ if and only if $d\phi(X) = X$. If $\gamma : I \to M$ is an arbitrary curve in M with $\phi(\gamma(t)) = \gamma(t)$ for all $t \in I$, then $d\phi(\frac{\nabla}{dt}\dot\gamma(t)) = \frac{\nabla}{dt}\dot\gamma(t)$ for all $t \in I$.

2. Let $\mathbb{H}^n := \{x \in \mathbb{R}^{n+1}\,,\ \langle\langle x, x\rangle\rangle = -1 \text{ and } x_0 > 0\}$ denote the n-dimensional hyperbolic space (see Exercise no. 4 in Sheet 5). We identify $T_x\mathbb{H}^n$ with $x^\perp := \{V \in \mathbb{R}^{n+1}\,,\ \langle\langle V, x\rangle\rangle = 0\}$. For V in $x^\perp$ we define $\|V\| := \sqrt{\langle\langle V, V\rangle\rangle}$. For $V \in x^\perp \setminus \{0\}$ show that
   $$\gamma_{x,V}(t) := \cosh(\|V\|t)x + \sinh(\|V\|t)\frac{V}{\|V\|}$$
   is a curve in $\mathbb{H}^n$.

3. Determine a linear map $\Phi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with $\Phi^*\langle\langle\,\cdot\,,\,\cdot\,\rangle\rangle = \langle\langle\,\cdot\,,\,\cdot\,\rangle\rangle$ such that its fixed point set is the plane spanned by $x \in \mathbb{H}^n$ and $V \in x^\perp \setminus \{0\}$. Show that its restriction to $\mathbb{H}^n$ defines an isometry $\phi : \mathbb{H}^n \to \mathbb{H}^n$. What is the fixed point set?

4. Conclude that $\frac{\nabla}{dt}\dot\gamma_{x,V}(t) = f(t)\dot\gamma_{x,V}(t)$ for all $t \in \mathbb{R}$ and for a suitable function f.

5. Show that $\gamma_{x,V}$ is a geodesic. *Hint: Calculate* $\|\dot{\gamma}_{x,V}(t)\|$. Are all non-constant geodesics in $\mathbb{H}^n$ of this form?

Exercise 4
Does there exist a Riemannian metric

1. on $\mathbb{R}^2$ such that all circles can be parametrized as geodesics?
2. on $\mathbb{R}^2 \setminus \{0\}$ such that all circles centered at 0 can be parametrized as geodesics?
3. on $\mathbb{R}^2 \setminus \{0\}$ such that all circles centered at 0 can be parametrized as geodesics but *no* ray through 0 is a geodesic?

Justify each of your answers.

Abgabe der Lösungen: ***Montag, den 3.12.2012*** *vor der Vorlesung.*

Universität Regensburg, Mathematik
Prof. Dr. Bernd Ammann
Dr. Nicolas Ginoux

WS 2012/13
3.12.2012

Differential Geometry I
Exercise Sheet no. 8

Exercise 1
Let (M^n, g) be a smooth n-dimensional Riemannian manifold and $f : M \to \mathbb{R}$ be a smooth function on M.

(a) Show that there exists a unique smooth tangent vector field – which we denote by $\mathrm{grad} f$ and call the *gradient vector field* of f – on M such that

$$\partial_X f = g(\mathrm{grad} f, X)$$

holds for every $X \in \mathfrak{X}(M)$.

(b) Given any $p \in M$ and any orthonormal basis $\{e_1, \dots, e_n\}$ of T_pM (i.e., $g_p(e_i, e_j) = \delta_i^j$), prove that $\mathrm{grad} f|_p = \sum_{j=1}^n \partial_{e_j} f \cdot e_j$.

(c) Given any chart $\varphi : U_\varphi \to V_\varphi$ of M, show that

$$\mathrm{grad} f_{|U_\varphi} = \sum_{i,j=1}^n g^{ij} \frac{\partial f}{\partial \varphi^i} \cdot \frac{\partial}{\partial \varphi^j},$$

where $(g^{ij})_{1 \le i,j \le n}$ is the inverse matrix of the matrix $(g_{kl})_{1 \le k,l \le n}$ of g in the chart φ.

Exercise 2 *(Poincaré half-space)*
Let $H := \{(x, y) \in \mathbb{R}^2 \, , \, y > 0\}$ denote the upper half-space and define the Riemannian metric g on H by

$$g := \frac{dx^2 + dy^2}{y^2}.$$

(a) Compute the Christoffel symbols of the Levi-Civita connection in the canonical coordinates x, y of H.

(b) Let $c : \mathbb{R} \to H$, $t \mapsto (t, 1)$ and $v_0 := (0, 1) \in \mathbb{R}^2 \cong T_{(0,1)}H$. Show that, for the parallel vector field v along c with $v(0) = v_0$, the vector $v(t)$ makes an angle equal to t with the y-axis, for all t.

Exercise 3 *(Surfaces of Revolution and Clairaut's theorem)*
For an interval I and a positive smooth function $f : I \to \mathbb{R}^+$ we define

$$F(x, \phi) := \begin{pmatrix} x \\ f(x)\cos(\phi) \\ f(x)\sin(\phi) \end{pmatrix}.$$

The surface of revolution generated by f is

$$M_f := \left\{ F(x, \phi) \,\middle|\, x \in I, \phi \in \mathbb{R} \right\}.$$

Then $\partial F / \partial x$ and $\partial F / \partial \phi$ define vector fields along $F : I \times \mathbb{R} \to M_f$.

(a) Construct vector fields X_x, $X_\phi \in \mathcal{X}(M_f)$ such that $X_x \circ F = \partial F/\partial x$ and $X_\phi \circ F = \partial F/\partial \phi$. Show that X_x and X_ϕ are everywhere orthogonal.

(b) Show that $c_{x_0} : \mathbb{R} \to M_f$, $\phi \mapsto F(x_0, \phi)$ is a geodesic if and only if $f'(x_0) = 0$. *Hint: Exercise 2 from sheet no. 7 is helpful.*

(c) Let $\gamma(t) = F(x(t), \phi(t))$ be a curve in M_f. Verify

$$g\Big(\dot{\gamma}(t), \frac{\partial F}{\partial \phi}(x(t), \phi(t))\Big) = f(x(t))^2 \dot{\phi}(t) \tag{1}$$

(d) Verify the formula

$$\frac{d}{dt}\left(g\Big(\dot{\gamma}(t), \frac{\partial F}{\partial \phi}\Big)\right) = g\Big(\frac{\nabla}{dt}\dot{\gamma}(t), \frac{\partial F}{\partial \phi}\Big) + \dot{\phi}(t) g\Big(\dot{\gamma}(t), \frac{\partial^2 F}{\partial^2 \phi}\Big) + \dot{x}(t) g\Big(\dot{\gamma}(t), \frac{\partial^2 F}{\partial \phi\, \partial x}\Big)$$

where we suppressed $(x(t), \phi(t))$ in the notation.

(e) Show that both $\|\dot{\gamma}(t)\|$ and $f(x(t))^2\dot{\phi}(t)$ are constant if γ is a geodesic. *Hint: Derive both sides of* (1) *with respect to* t *and use (d).*

(f) (Extra question, 2 bonus points.) Does "only if" hold in (e)?

Exercise 4

Let ∇ be any connection on a smooth manifold M. Let $c : I \to M$ be any smooth curve and denote by $P_{s,t} : T_{c(s)}M \to T_{c(t)}M$ the parallel transport along c. Show that, for any smooth vector field X along c and any $t_0 \in I$, we have

$$\frac{\nabla X}{dt}(t_0) = \lim_{t \to t_0} \left(\frac{P_{t,t_0}(X(t)) - X(t_0)}{t - t_0} \right).$$

Abgabe der Lösungen: ***Montag, den 10.12.2012*** *vor der Vorlesung.*

Universität Regensburg, Mathematik
Prof. Dr. Bernd Ammann
Dr. Nicolas Ginoux

WS 2012/13
10.12.2012

Differential Geometry I
Exercise Sheet no. 9

Exercise 1
Let $\gamma : [a,b] \longrightarrow M$ be a piecewise C^1 curve on a smooth Riemannian manifold (M,g).

(a) Prove that $L[\gamma]^2 \leq 2(b-a) \cdot E[\gamma]$, where $E[\gamma] := \frac{1}{2}\int_a^b g(\dot\gamma,\dot\gamma)dt$ is the *energy* of the curve γ.

(b) Show that $L[\gamma]^2 = 2(b-a)\cdot E[\gamma]$ holds iff γ is parametrized proportionally to arc-length.

Exercise 2
Let M be a smooth submanifold of $\mathbb{R}^k$.

(a) Show that, if M is closed, then M is complete.

(b) Show that the converse statement is wrong.

Exercise 3
Let (M,g) be a connected complete non-compact Riemannian manifold and $p \in M$ be a point.

(a) Show that there exists a sequence $(p_i)_{i\in\mathbb{N}}$ in M such that $d(p,p_i) \underset{i\to\infty}{\longrightarrow} \infty$.

(b) Show that, for each $i \in \mathbb{N}$, there exist $X_i \in T_pM$ and $r_i \in [0,\infty[$ with $g_p(X_i,X_i) = 1$ and $p_i = \exp_p(r_i X_i)$.

(c) Show that the sequence $\{X_i\}_{i\in\mathbb{N}}$ admits a converging subsequence and deduce that there exists a ray $\gamma : [0,\infty) \to M$ in (M,g) with $\gamma(0) = p$.

Exercise 4
Let M be a connected m-dimensional manifold, and assume that $N \subset M$ is an n-dimensional submanifold, i.e. for every $p \in N$ there is a chart $\phi : U \to V \subset \mathbb{R}^m$, $p \in U$ such that $\phi(U \cap N) = V \cap (\mathbb{R}^n \times \{0\})$. Let g be a Riemannian metric on M, such that (M,g) is complete, and assume that N is a closed (as a subset of M). Fix a point $q \in M$.

(a) Show the existence of a point $p \in N$ with $d(q,p) = d(q,N)$, where $d(q,N) := \inf_{x\in N}\{d(q,x)\}$. Is p unique? Justify your answer.

(b) Prove that there is a geodesic γ from q to p with length $L[\gamma] = d(q,p)$.

(c) Show that γ meets N orthogonally.

Abgabe der Lösungen: ***Montag, den 17.12.2012*** *vor der Vorlesung.*

Universität Regensburg, Mathematik
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WS 2012/13
17.12.2012

Differential Geometry I
Exercise Sheet no. 10

Exercise 1
Show that the map $p : S^n \to \mathbb{R}\mathrm{P}^n$, $x \mapsto \mathbb{R}x$ is a local diffeomorphism, i.e. every $x \in S^n$ is in an open set U such that $p(U)$ is open in $\mathbb{R}\mathrm{P}^n$ and such that $p|_U$ is a diffeomorphism from U to $p(U)$. Show that $\mathbb{R}\mathrm{P}^n$ carries a metric g_0 such that p^*g_0 is the standard metric on S^n. This metric g_0 is called the *standard metric of* $\mathbb{R}\mathrm{P}^n$. Determine the injectivity radius of $(\mathbb{R}\mathrm{P}^n, g_0)$.

Exercise 2
The *tautological bundle* on the n-dimensional real projective space $\mathbb{R}\mathrm{P}^n$ is given by $L := \{(\ell, y) \in \mathbb{R}\mathrm{P}^n \times \mathbb{R}^{n+1}, y \in \ell\}$ together with the projection map $\pi : L \to \mathbb{R}\mathrm{P}^n$, $(\ell, y) \mapsto \ell$. Prove that there does not exist any continuous and nowhere vanishing section s of $\pi : L \to \mathbb{R}\mathrm{P}^n$. *(Hint: Interprete such a section as a map* $\mathbb{R}\mathrm{P}^n \to \mathbb{R}^{n+1} \setminus \{0\}$*; considering the composition with the map* $S^n \to \mathbb{R}\mathrm{P}^n$*, get a map* $S^n \to S^n$ *which has to be* $\pm\mathrm{Id}$*; conclude.)*

Exercise 3 *(Geodesics and distance function on products)*

(a) Let $\gamma : [a, b] \to M$ be a piecewise C^1 curve on a smooth Riemannian manifold (M, g). Prove that γ minimizes the energy functional $E : c \mapsto \frac{1}{2}\int_a^b g(\dot{c}, \dot{c})\,dt$ among all piecewise C^1 curves $c : [a, b] \to M$ with $c(a) = p$ and $c(b) = q$ iff γ is a minimal geodesic.

(b) From now on let $(M, g) := (M_1 \times M_2, g_1 \oplus g_2)$, where (M_i, g_i) is a smooth Riemannian manifold and the product manifold $M_1 \times M_2$ (see Exercise no. 1 in Sheet 2) is equipped with the *product metric* $g_1 \oplus g_2$, which is defined at $p = (p_1, p_2) \in M_1 \times M_2$ by:

$$(g_1 \oplus g_2)|_{(p_1,p_2)}\left((X_1, X_2), (Y_1, Y_2)\right) := g_1|_{p_1}(X_1, Y_1) + g_2|_{p_2}(X_2, Y_2)$$

for all $X_i, Y_i \in T_{p_i}M$, $i = 1, 2$. Show that, if $\gamma_i : [a, b] \to M_i$ is a piecewise C^1 curve, $i = 1, 2$, then $\gamma := (\gamma_1, \gamma_2) : [a, b] \to M_1 \times M_2$ is a piecewise C^1 curve with $E(\gamma) = E(\gamma_1) + E(\gamma_2)$.

(c) Show that γ is a minimal geodesic iff γ_1 and γ_2 are minimal geodesics.

(d) Deduce that the distance function d associated to $g = g_1 \oplus g_2$ is given by

$$d\left((x_1, x_2), (y_1, y_2)\right) = \sqrt{d_1(x_1, y_1)^2 + d_2(x_2, y_2)^2},$$

for all $(x_1, x_2), (y_1, y_2) \in M_1 \times M_2$, where d_i is the distance function associated to the metric g_i on M_i.

Exercise 4 *(Sufficient criterion for the existence of a line)*
Let (M, g) be a complete smooth Riemannian manifold.

(a) Let $(X_k)_{k\in\mathbb{N}}$ be a sequence in TM converging to some X and $a, b \in \mathbb{R}$ with $a < b$. Show that, if $\gamma_{X_k|_{[a,b]}} : [a, b] \to M$ is a shortest curve, then so is $\gamma_{X|_{[a,b]}} : [a, b] \to M$. Here and as usual, for any $Y \in TM$, we denote by $\gamma_Y : \mathbb{R} \to M$ the unique geodesic with $\gamma_Y(0) = \pi(Y) \in M$ and $\dot{\gamma}_Y(0) = Y$.

(b) Assume the existence of two sequences $(x_k)_{k\in\mathbb{N}}$, $(y_k)_{k\in\mathbb{N}}$ in M, of a point $p \in M$ and of an $R \in]0, \infty[$ with $d(x_k, p) \underset{k\to\infty}{\longrightarrow} \infty$, $d(y_k, p) \underset{k\to\infty}{\longrightarrow} \infty$ and such that every shortest curve from x_k to y_k meets the ball $B_R(p)$. Show that there exists a line in (M, g).
(Hint: construct a limit of a sequence of shortest curves. Exercise no. 3 of Sheet 9 may be helpful.)

Abgabe der Lösungen: ***Montag, den 7.1.2013*** *vor der Vorlesung.*

Wir wünschen allen Teilnehmerinnen und Teilnehmern frohe Weihnachten und einen guten Rutsch!

Universität Regensburg, Mathematik
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WS 2012/13
7.01.2013

Differential Geometry I
Exercise Sheet no. 11

Exercise 1

(a) Let $V \to M$ be a complex vector bundle endowed with a Hermitian metric $\langle \cdot\,, \cdot \rangle$ over a smooth manifold. Show that the dual bundle $V^* \to M$ is isomorphic to the so-called *conjugate* vector bundle $\overline{V} \to M$, where $\overline{V}_x := V_x$ but where $\lambda \cdot v := \overline{\lambda} v$ for all $v \in V_x$, $x \in M$ and $\lambda \in \mathbb{C}$.

(b) Let $\tau \to \mathbb{C}\mathrm{P}^n$ be the tautological bundle as defined in Exercise 2 of Sheet no. 10. Using the canonical Hermitian inner product on $\mathbb{C}^{n+1}$, construct a Hermitian metric on τ.

Exercise 2

Let ∇ be any connection on the tangent bundle TM of a smooth manifold M and T be its torsion, that is, $T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]$ for all $X, Y \in \mathfrak{X}(M)$. Show that T is a tensor on M, more precisely show that T defines a section of the vector bundle $T^*M \otimes T^*M \otimes TM \to M$.

Exercise 3

Let M be any smooth manifold.

(a) Given any vector bundles $E \to M$ and $F \to M$ with connections ∇^E and ∇^F respectively, prove that there exists a unique connection ∇ on the tensor product bundle $E \otimes F \to M$ such that $\nabla(s \otimes s') = (\nabla^E s) \otimes s' + s \otimes (\nabla^F s')$ for all sections s of E and s' of F.

(b) Let $E \to M$ be a vector bundle with connection ∇^E. In each fiber E_p the trace tr_p is an element of $\mathrm{Hom}_{\mathbb{K}}(E_p^* \otimes E_p, \mathbb{K}) \cong E_p \otimes E_p^*$. Show that $p \mapsto \mathrm{tr}_p$ is a smooth map from M to $E \otimes E^*$. Show that it is a parallel section of $E \otimes E^* \to M$.

(c) Given any real vector bundle $E \to M$ with Riemannian metric $\langle \cdot\,, \cdot \rangle$ and connection ∇^E, show that the Riemannian metric – as a section of the vector bundle $E^* \otimes E^* \to M$ – is parallel iff the connection ∇^E is metric.

Exercise 4

Let $V \to M$ be a real or complex line bundle over a smooth manifold. Show that the tensor vector bundle $V^* \otimes V \to M$ is trivial.

Abgabe der Lösungen: ***Montag, den 14.1.2013*** *vor der Vorlesung.*

Wir wünschen allen Teilnehmerinnen und Teilnehmern ein erfolgreiches neues Jahr!

Universität Regensburg, Mathematik
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WS 2012/13
14.01.2013

Differential Geometry I
Exercise Sheet no. 12

Exercise 1 Let (M^n, g) be a Riemannian manifold and (U_φ, φ) be a chart on M.

(a) Show that the components $R^l_{ijk} := d\varphi^l\left(R(\frac{\partial}{\partial\varphi^i}, \frac{\partial}{\partial\varphi^j})\frac{\partial}{\partial\varphi^k}\right)$ of the curvature tensor R of g on U_φ are given in terms of the Christoffel symbols associated to (U_φ, φ) by

$$R^l_{ijk} = \frac{\partial\Gamma^l_{kj}}{\partial\varphi^i} - \frac{\partial\Gamma^l_{ki}}{\partial\varphi^j} + \sum_{m=1}^n \left(\Gamma^l_{mi}\Gamma^m_{kj} - \Gamma^l_{mj}\Gamma^m_{ki}\right).$$

(b) Deduce that, if $\mathrm{ric} = \sum_{i,j=1}^n \mathrm{ric}_{ij} d\varphi^i \otimes d\varphi^j$ denotes the decomposition of the Ricci-tensor of g, then

$$\mathrm{ric}_{ij} = \sum_{k=1}^n R^k_{kij} = \sum_{k=1}^n \left(\frac{\partial\Gamma^k_{ji}}{\partial\varphi^k} - \frac{\partial\Gamma^k_{jk}}{\partial\varphi^i} + \sum_{m=1}^n \left(\Gamma^k_{mk}\Gamma^m_{ji} - \Gamma^k_{mi}\Gamma^m_{jk}\right)\right).$$

Exercise 2
Let $(M := M_1 \times M_2, g := g_1 \oplus g_2)$ be the product of two Riemannian manifolds as in Exercise 3.(b) of Sheet no. 10.

(a) Show that the Levi-Civita connection ∇ of (M, g) is given by

$$\nabla_{(X_1,X_2)}(Y_1, Y_2) = \nabla^{M_1}_{X_1} Y_1 + \nabla^{M_2}_{X_2} Y_2 + \partial_{X_1} Y_2 + \partial_{X_2} Y_1,$$

for all $X_1, Y_1 \in \Gamma(\pi_1^* TM_1)$, $X_2, Y_2 \in \Gamma(\pi_2^* TM_2)$ and where $\partial_{X_1} Y_2$ (resp. $\partial_{X_2} Y_1$) denotes the usual derivative (make sense of this).

(b) Deduce that the curvature tensor R of (M, g) is given by

$$R^M_{X,Y} Z = R^{M_1}_{X_1,Y_1} Z_1 + R^{M_2}_{X_2,Y_2} Z_2,$$

for all $X_i, Y_i, Z_i \in T_{x_i} M_i$, $i = 1, 2$, where $X := X_1 + X_2$, $Y := Y_1 + Y_2$ and $Z := Z_1 + Z_2$. (Here we write $R_{X,Y}$ instead of $R(X, Y)$ as this is better for typesetting in this context.)

(c) Calculate the Ricci tensor and the scalar curvature of M in terms of the Ricci tensor and the scalar curvature of M_1 and M_2.

Exercise 3
Let $E, F \to M$ be (real or complex) vector bundles with connections ∇^E, ∇^F over a given manifold M and $x \in M$ be a point. Prove the following identities:

(a) The curvature tensor of the connection $\nabla^E \oplus \nabla^F$ on $E \oplus F \to M$ is given by

$$R^{E\oplus F}_{X,Y} = R^E_{X,Y} \oplus R^F_{X,Y},$$

for all $X, Y \in T_xM$.

(b) The curvature tensor of the tensor connection on $E \otimes F \to M$ as defined in Exercise 3 of Sheet no. 11 is given by

$$R^{E\otimes F}_{X,Y} = R^E_{X,Y} \otimes \mathrm{Id}_F + \mathrm{Id}_E \otimes R^F_{X,Y},$$

for all $X, Y \in T_xM$.

(c) The curvature tensor of the dual bundle $E^* \to M$ endowed with the induced connection is given by

$$(R^{E^*}_{X,Y}\alpha)(V) = -\alpha(R^E_{X,Y}V),$$

for all $X, Y \in T_xM$, for all $V \in E_x$ and $\alpha \in E^*_x$.

Exercise 4

Let (M^n, g) be a Riemannian manifold and denote by ∇ resp. R the Levi-Civita connection resp. the Riemannian curvature tensor of (M^n, g). Let the $(0,4)$-tensor $\tilde{R}$ be defined by $\tilde{R}(X, Y, Z, W) := g(R(X, Y)Z, W)$, for all $X, Y, Z, W \in T_xM$ and $x \in M$.

(a) Let $x \in M$ be a point. Prove that the following identities are satisfied: for all $X, Y, Z, T, U \in T_pM$,

$$(\nabla_X \tilde{R})(Y, Z, T, U) = -(\nabla_X \tilde{R})(Z, Y, T, U) = -(\nabla_X \tilde{R})(Y, Z, U, T) = (\nabla_X \tilde{R})(T, U, Y, Z).$$

(b) For a given tensor field $A \in \Gamma(T^*M \otimes T^*M)$ let the *divergence* of A be defined by

$$\mathrm{div}(A)(X) := \sum_{j=1}^{n} (\nabla_{E_j} A)(E_j, X) \qquad \forall X \in TM,$$

where $\{E_j\}_{1\le j\le n}$ is a local orthonormal basis of TM, that is, $g(E_i, E_j) = \delta_i^j$. Prove using the second Bianchi identity:

$$\mathrm{div}(\mathrm{ric}) = \frac{1}{2} d\,\mathrm{scal}.$$

(Hint: for a given point $x \in M$, the basis $\{E_j\}_{1\le j\le n}$ can be chosen such that $(\nabla E_i)_{|x} = 0$ holds; how can this be done?)

(c) *Application*: Assuming $n \geq 3$, the manifold M connected and the existence of a smooth function $f : M \to \mathbb{R}$ with $\mathrm{ric} = f \cdot g$ on M, prove that f is constant on M. Such a Riemannian manifold is then called *Einstein*.

Abgabe der Lösungen: ***Montag, den 21.1.2013*** *vor der Vorlesung.*

Universität Regensburg, Mathematik
Prof. Dr. Bernd Ammann
Dr. Nicolas Ginoux

WS 2012/13
21.01.2013

Differential Geometry I
Exercise Sheet no. 13

Exercise 1
Let (M^n, g) be a connected n-dimensional Riemannian manifold. Assume $n \geq 3$ and that, for each $p \in M$ and any two planes $E, E' \subset T_pM$, we have $K(E) = K(E')$, where $K(E)$ denotes the sectional curvature of the plane E.

(a) Show the existence of a smooth function $\kappa : M \to \mathbb{R}$ such that, for all $X, Y, Z, T \in \mathfrak{X}(M)$,

$$\langle R(X,Y)Z, T\rangle = \kappa \cdot \Big(g(Y,Z)(gX,T) - g(X,Z)g(Y,T) \Big)$$

holds on M.

(b) Deduce that $\mathrm{ric} = (n-1)\kappa g$ and that (M^n, g) has constant sectional curvature, i.e. that κ is constant.

Exercise 2 *(Möbius strip)*
Let $F : \mathbb{R} \times]-1,1[\to \mathbb{R}^3$ be the map defined by

$$F(x,y) := \begin{pmatrix} (1 + \frac{y}{2}\cos(\frac{x}{2}))\cos(x) \\ (1 + \frac{y}{2}\cos(\frac{x}{2}))\sin(x) \\ \frac{y}{2}\sin(\frac{x}{2}) \end{pmatrix}$$

and let $M := F(\mathbb{R} \times]-1,1[) \subset \mathbb{R}^3$.

(a) Show that M is a smooth 2-dimensional submanifold of $\mathbb{R}^3$.

(b) Show that, for every $(x,y) \in \mathbb{R} \times]-1,1[$, the vector $\frac{\frac{\partial F}{\partial x}(x,y) \times \frac{\partial F}{\partial y}(x,y)}{\|\frac{\partial F}{\partial x}(x,y) \times \frac{\partial F}{\partial y}(x,y)\|} \in \mathbb{R}^3$ has unit norm and is orthogonal to $T_{F(x,y)}M$. Here "$\times$" denotes the cross product for vectors in $\mathbb{R}^3$.

(c) Show that no continuous unit normal field exists on M and deduce that M is not orientable.

Exercise 3
Let $C := \{x = (x_0, x_1, x_2, x_3, x_4) \in \mathbb{R}^5 \,|\, x_1^2 + x_2^2 + x_3^2 + x_4^2 = \cosh(x_0)^2\}$. Compute the second fundamental form (in $\mathbb{R}^5$), the Ricci-tensor and the scalar curvature of (M, g), where g is the Riemannian metric induced by the standard Euclidean inner product.
Hint: Show that the second fundamental form has pointwise two eigenvalues, κ with multiplicity 1 and $-\kappa$ with multiplicity 3.

Exercise 4
Let $M \subset \widehat{M}$ be a submanifold of the Riemannian manifold $\widehat{M}$. It is called totally geodesic iff $\mathbb{I} \equiv 0$.

(a) Show that M is a totally geodesic iff every geodesic of M is also a geodesic of $\hat{M}$.

(b) Assume additionally that M is complete. Show that M is totally geodesic iff every geodesic $\gamma : I \to \hat{M}$, $0 \in I$, of $\hat{M}$ with $\dot{\gamma}(0) \in TM$ is contained in M.

Abgabe der Lösungen: ***Montag, den 28.1.2013*** *vor der Vorlesung.*

Universität Regensburg, Mathematik
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Dr. Nicolas Ginoux

WS 2012/13
28.01.2013

Differential Geometry I
Exercise Sheet no. 14

Exercise 1
Let (M^n, g) be a connected Riemannian manifold and $f : M \to M$ be an isometry. Show that, if at a point $p \in M$, one has $f(p) = p$ and $d_p f = \mathrm{Id}_{T_pM}$, then $f = \mathrm{Id}_M$.

Exercise 2
We denote as usual by $\mathbb{RP}^n$ the n-dimensional real projective space.

(a) Given any $p \in S^n$, we define $\omega \in \Gamma(\Lambda^n T_p^* S^n)$ as $\omega_p(V_1, V_2, \ldots, V_n) = \det(p, V_1, \ldots, V_n)$. Show that ω defines a trivialization of $\Lambda^n T_p^* S^n$ and an orientation of S^n.

(b) Show $(-\mathrm{Id}_{S^n})^* \omega = (-1)^{n+1} \omega$. In other words, show that

$$\omega_{-p}(d(-\mathrm{Id}_{S^n})(V_1), d(-\mathrm{Id}_{S^n})(V_2), \ldots, d(-\mathrm{Id}_{S^n})(V_n)) = (-1)^{n+1} \omega_p(V_1, V_2, \ldots, V_n)$$

for all $p \in S^n$ and all $V_1, \ldots, V_n \in T_p S^n$.

(c) Deduce that $\mathbb{RP}^n$ is orientable iff n is odd.
(Hint: The proof that it is non-orientable for n even can be carried out similarly to Exercise 2 of Sheet no. 10.)

Exercise 3
Let M be a compact (orientable) surface in $\mathbb{R}^3$. Let $\overline{B}_r(0)$ be the closed ball of radius r around 0 in $\mathbb{R}^3$, and let $S_r(0) = \partial \overline{B}_r(0)$ be its boundary.

(a) Show that the infimum $R := \inf\{r > 0 \,|\, M \subset \overline{B}_r(0)\} > 0$ is attained, and conclude that $M \cap S_R(0)$ is not empty.

(b) Show that $T_x M$ is the orthogonal complement of x for any $x \in M \cap S_R(0)$. Show for any such x, that the two principal curvatures in x have the same sign (for a given choice of a unit normal field).

(c) Deduce that the Gauss curvature of M is positive in any $x \in M \cap S_R(0)$.

(d) Are there compact minimal surfaces M in $\mathbb{R}^3$? Justify your answer. *(We want to recall the fact that a surface is called minimal if the mean curvature H vanishes on all of M.)*

Exercise 4
Let $M \subset \mathbb{R}^3$ be a compact and connected surface. We accept without proof that $\mathbb{R}^3 \setminus M$ has two connected components, one bounded and one unbounded.

(a) Show the existence of a smooth map $N : M \to S^2$ with $N(p) \perp T_pM$ for all $p \in M$.

(b) Deduce that M is orientable.

(c) Show that N is surjective.
(Hint: show that any 2-dimensional vector subspace of $\mathbb{R}^3$ is the tangent space of M at two or more points of M.)
Comment: Such a map N is called "the" Gauß map of the surface. It is unique up to a sign.

Abgabe der Lösungen: ***Montag, den 4.2.2013*** *vor der Vorlesung.*

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Differential Geometry I
Exercise Sheet no. 15

Exercise 1
Right or wrong? Justify shortly each of your answers. In the whole exercise, V and W denote smooth $\mathbb{K}$-vector bundles over a smooth manifold M, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

(a) If V and W are trivial, then so are $V \oplus W$, $V \otimes W$, $\mathrm{Hom}(V, W)$ and $\Lambda^2 V$.

(b) For any $n \geq 3$, there exists an n-dimensional compact smooth manifold whose tangent bundle is trivial.

(c) Each complex vector bundle with connection over a 1-dimensional manifold is flat.

(d) Each real vector bundle over S^1 is trivial.

(e) If V and $V \oplus W$ are trivial, then so is W.

(f) Every complex line bundle with connection which admits a nowhere-vanishing parallel section is flat.

(g) Every real line bundle with connection is flat.

(h) There exists on every manifold a vector bundle with flat connection.

(i) The tangent bundle of every Riemannian manifold admits a unique metric connection.

(j) If a surface $M \subset \mathbb{R}^3$ contains a line (i.e., if there exists $p, v \in \mathbb{R}^3$, $v \neq 0$, with $p + \mathbb{R} \cdot v \subset M$), then $K \leq 0$ along that line.

Exercise 2
Let $M := S^2 \times \mathbb{R} \subset \mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^4$. Compute (after choosing a smooth unit normal field) the Weingarten map, the mean curvature, the Ricci-tensor and the scalar curvature of M.

Exercise 3
Let M and M' be compact surfaces.

(a) Assume M and M' to carry metrics with negative sectional (or Gauß) curvature. Does the connected sum $M \sharp M'$ carry such a metric? Justify your answer.

(b) Same question if M and M' carry a metric with positive sectional curvature.

Exercise 4
Let $V \to M$ be a real or complex vector bundle over an arbitrary smooth manifold. Show that there is a scalar product on V.
Hint: Construct at first a scalar product on trivial vector bundles. Then use local trivializations and a partition of unity (Lemma IV.6.2) to prove the general case.

Exercise 5
Given $n \in \mathbb{N}$, $n \geq 2$, let $\pi_N : S^n \setminus \{e_{n+1}\} \to \mathbb{R}^n$, $x \mapsto \frac{1}{1-x_{n+1}}(x_1, \ldots, x_n)$ and $\pi_S : S^n \setminus \{-e_{n+1}\} \to \mathbb{R}^n$, $x \mapsto \frac{1}{1+x_{n+1}}(x_1, \ldots, x_n)$, denote the stereographic projections from the North and South pole respectively. Recall the definition $f_*X = df \circ X \circ f^{-1}$ for a diffeomorphism $f : M \to N$ and a vector field X on M.

(a) Given $v \in \mathbb{R}^n \setminus \{0\}$, define the vector field X on $S^n \setminus \{e_{n+1}\}$ by $X := \pi_N^* v$, that is, $X_{|_p} := (d_p\pi)^{-1}(v)$ for all $p \in S^n \setminus \{e_{n+1}\}$. Show that $((\pi_S)_*X)_{|_x} = |x|^2 \cdot \left(v - 2\langle v, \frac{x}{|x|}\rangle \frac{x}{|x|}\right)$ for all $x \in \mathbb{R}^n \setminus \{0\}$.

(b) Deduce that X can be extended uniquely as a smooth vector field $\widetilde{X}$ on S^n, and determine its zero points.

(c) From now on let $n = 2$. Apply the Poincaré-Hopf theorem to compute the index of $\widetilde{X}$ in e_{n+1}.

(d) In the case $n = 2$ check the result of (c) by a direct computation using part (a).

Abgabe der Lösungen: ***Montag, den 11.2.2013*** *bei Nicolas Ginoux (M122).*