SPECTRAL PROPERTIES OF THE DIRAC OPERATOR AND GEOMETRICAL STRUCTURES

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ABSTRACT. These lectures aim to give an elementary expository on basic results about the first eigenvalue of the Dirac operator, on compact Riemannian Spin manifolds with positive scalar curvature.

For this, we select some key ingredients which illustrate the basic objects and some of their properties as Clifford algebras, spin groups, connections, covariant derivatives, Dirac and Twistor operators.

We end by pointing out how the size of the gap around zero in the spectrum of the Dirac operator, increases when the geometrical structure is Kähler or Quaternion-Kähler. We refer to [BHMM] for an extensive and recent study of Conformal Spin Geometry.

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1. Introduction

The Dirac operator (square root of the Laplacian) is a first order, elliptic differential operator which plays a central role in Mathematical Physics, Topology and Geometry. The Atiyah–Singer Index Theorem, the Lichnerowicz Theorem and the Seiberg–Witten Invariants illustrate the importance of this operator in Physics as well as in Mathematics.

Unlike differential forms, spinor fields depend on the geometrical and topological data, hence the subtlety of natural operators acting on them.

The aim of these lectures is to give an elementary introduction to the subject starting with the building blocks of Spin Geometry such as Clifford Algebras, Spin groups and Spinors, in order to prove some recent results concerning the spectral geometry of the Dirac operator.

For convenience, we introduce basic ingredients in Riemannian Geometry as vector and principal bundles as well as the notion of connection form and the associated covariant derivative.

Attached to the Dirac operator is the twistor operator, the complement of the Dirac operator, which is relevant in this setup. We explicitely define these operators in terms of projections on vector spaces. We show, that there is a gap around zero in the spectrum of the Dirac operator on compact Riemannian spin manifolds and that the width of this gap depends on the geometry of the manifold.

A classical reference for basic notions on Spin Geometry is the famous book of Blaine Lawson and Marie-Louise Michelsohn [LM], however in these notes we give a brief presentation of the necessary material needed to illustrate some basic results on the first eigenvalues of the Dirac operator.

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2. Clifford Algebras, Spin Groups and their Representations

The aim of this section is to present the algebraic ingredients lying at the heart of spin geometry. First we introduce Clifford algebras, which are algebras naturally associated to symmetric bilinear forms on vector spaces. For simplicity, we restrict ourselves to complex Clifford algebras and their representations. In the Clifford algebra of the n-dimensional euclidian space, we construct the spin group Spin_n, which is a 2-fold covering of the n-dimensional special orthogonal group, SO_n.

2.1. **Clifford Algebras.**

Definition 2.1. Let V be a vector space over a field K of dimension n and g a non-degenerate bilinear form on V. The *Clifford algebra* $Cl(V,g)$ associated to g on V is an associative algebra with unit, defined by

$$Cl(v,g) := T(V)\big/I(V,g)$$

where $T(V)$ is the tensor algebra of V, and $I(V,g)$ the ideal generated by all elements of the form $x \otimes x + g(x,x)1$, for $x \in V$.

Remarks 2.2. (1) The Clifford algebra $Cl(V,g)$ is the algebra generated by V with the relation $x \cdot y + y \cdot x = -2g(x,y)1$, for $x, y \in V$.

(2) If $(e_1, \ldots, e_n)$ is a g–orthonormal basis of V, then

$$\{\, e_{i_1} \cdot \ldots \cdot e_{i_k} \mid 1 \leq i_1 < \ldots < i_k \leq n, \quad 0 \leq k \leq n \,\}$$

is a basis of $Cl(V,g)$, thus $\dim Cl(V,g) = 2^n$.

(3) There is a canonical isomorphism of vector spaces, between the exterior algebra and the Clifford algebra of (V,g) which is given by:

$$\begin{array}{ccc} \Lambda^* V & \xrightarrow{\simeq} & Cl(V,g) \\ e_{i_1} \wedge \ldots \wedge e_{i_k} & \longmapsto & e_{i_1} \cdot \ldots \cdot e_{i_k}. \end{array}$$

This isomorphism does not depend on the choice of the basis.

Examples 2.3. Let Cl_n denote the real Clifford algebra associated to the canonical scalar product $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$ on $\mathbb{R}^n$ and $(e_1, \ldots, e_n)$ the canonical orthonormal basis of $\mathbb{R}^n$. Then

– a basis of Cl_1 is given by $\{1, e_1\}$. Since $e_1^2 = -1$, one has

$$Cl_1 \simeq \mathbb{C}.$$

– a basis of Cl_2 is given by $\{1,\, e_1,\, e_2,\, e_1 \cdot e_2\}$. Since $e_1^2 = e_2^2 = (e_1 \cdot e_2)^2 = -1$, one has

$$Cl_2 \simeq \mathbb{H}.$$

– a basis of Cl_3 is given by $\{1, e_1,\, e_2,\, e_3,\, e_1{\cdot}e_2,\, e_2{\cdot}e_3,\, e_3{\cdot}e_1,\, e_1{\cdot}e_2{\cdot}e_3\}$. One can easily check that, with the help of the following natural linear identifications with the quaternions:

$$\{1,\, e_1 \cdot e_2,\, e_2 \cdot e_3,\, e_3 \cdot e_1\} \simeq \{1, i, j, k\}$$

and

$$\{e_1, e_2, e_3, e_1 \cdot e_2 \cdot e_3\} \simeq \{i, j, k, -1\}$$

leads to

$$Cl_3 \simeq \mathbb{H} \oplus \mathbb{H}\,.$$

Proposition 2.4 (Universal property)**.** *Let A be an associative algebra with unit and $f\colon V \to A$ a linear map such that for all $v \in V$*

$$f(v)^2 = -g(v,v)1.$$

Then f uniquely extends to a K–algebra homomorphism

$$\widetilde{f}\colon Cl(V,g) \to A.$$

Furthermore, $Cl(V,g)$ is the unique associative K–algebra with this property.

Corollary 2.5. *Let $f\colon (V,g) \to (\overline{V},\overline{g})$ be an isometry between euclidian vector spaces. Then f uniquely extends to a K–algebra homomorphism*

$$\widetilde{f}\colon Cl(V,g) \to Cl(\overline{V},\overline{g}).$$

Remark 2.6. We denote by $\mathrm{O}(V,g)$ the space of isometric homomorphisms of an euclidian vector space (V,g). On the Clifford algebra $Cl(V,g)$, we have two useful endomorphisms:

(1) The isometry $-\operatorname{Id} \in \mathrm{O}(V,g)$ gives rise to

$$\begin{array}{rccc} \alpha\colon & Cl(V,g) & \longrightarrow & Cl(V,g) \\ & e_{i_1}\cdot\ldots\cdot e_{i_k} & \longmapsto & (-1)^k e_{i_1}\cdot\ldots\cdot e_{i_k} \end{array}$$

As $\alpha^2 = \operatorname{Id}$, we get the decomposition

$$Cl(V,g) = Cl^0(V,g) \oplus Cl^1(V,g),$$

where $Cl^i(V,g) := \{\,\varphi \in Cl(V,g) \mid \alpha(\varphi) = (-1)^i\varphi\,\}$. Clearly, we have $Cl^i(V,g)\cdot Cl^j(V,g) \subset Cl^{i+j}(V,g)$ for $i,\, j \in \mathbb{Z}_2$, thus the Clifford algebra $Cl(V,g)$ is a $\mathbb{Z}_2$–graded algebra. The subspace $Cl^0(V,g)$ is called the *even* part, $Cl^1(V,g)$ the *odd* part of $Cl(V,g)$.

(2) Let ${}^t(\;)\colon T(V) \to T(V)$ be the K–algebra homomorphism defined by ${}^t(x_{i_1}\otimes\ldots\otimes x_{i_k}) = x_{i_k}\otimes\ldots\otimes x_{i_1}$. Since ${}^t(I(V,g)) \subset I(V,g)$, there is an induced homomorphism

$$\begin{array}{rccc} {}^t(\;)\colon & Cl(V,g) & \longrightarrow & Cl(V,g) \\ & x_{i_1}\cdot\ldots\cdot x_{i_k} & \longmapsto & x_{i_k}\cdot\ldots\cdot x_{i_1} \end{array}$$

This map is called the *transposition.*

Another consequence of the universal property of Clifford algebras is the following:

Proposition 2.7. *The Clifford algebra Cl_n is isomorphic to the even part of Cl_{n+1}, i.e.,*

$$Cl_n \simeq Cl^0_{n+1}\,.$$

Proof. Let $\{e_1, \dots, e_n\}$ and $\{e_1, \dots, e_{n+1}\}$ denote the canonical basis of $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$ - with the obvious identification - and define the linear map

$$\begin{array}{rcl} f\colon \mathbb{R}^n & \longrightarrow & Cl^0_{n+1} \\ e_i & \longmapsto & e_i \cdot e_{n+1}. \end{array}$$

By definition of the function f, we have $f(e_i)^2 = -1$, thus f extends to $\widetilde{f}\colon Cl_n \to Cl^0_{n+1}$. Clearly, $\widetilde{f}$ is an injective linear map between vector spaces of the same dimension, thus the map $\widetilde{f}$ is an isomorphism. □

In remark 2.2, we have seen that there is a canonical isomorphism of vector spaces between Cl_n and $\Lambda^*\mathbb{R}^n$. The following proposition gives the relation between the multiplication in the Clifford algebra in terms of the exterior and interior products in the exterior algebra.

Proposition 2.8. *For all $v \in \mathbb{R}^n$ and all $\varphi \in Cl_n$, we have*

$$v \cdot \varphi \simeq v \wedge \varphi - v \lrcorner \varphi \quad \textit{and} \quad \varphi \cdot v \simeq (-1)^p (v \wedge \varphi + v \lrcorner \varphi),$$

where $\wedge$ denotes the exterior, $\lrcorner$ the interior product and $\varphi \in \Lambda^p\mathbb{R}^n \subset \Lambda^*\mathbb{R}^n \simeq Cl_n$.

Proof. Let $v = e_j$ and $\varphi = e_{i_1} \cdot \ldots \cdot e_{i_p}$.

1. If there exists i_k such that $j = i_k$ then $v \wedge \varphi = 0$ and

$$\begin{aligned} v \lrcorner \varphi &= (-1)^{k-1} e_{i_1} \wedge \ldots \wedge e_{i_{k-1}} \wedge e_{i_{k+1}} \wedge \ldots \wedge e_{i_p} \\ &\simeq (-1)^{k-1} e_{i_1} \cdot \ldots \cdot e_{i_{k-1}} \cdot e_{i_{k+1}} \cdot \ldots \cdot e_{i_p} \\ &= -v \cdot \varphi \\ &= (-1)^p \varphi \cdot v. \end{aligned}$$

2. If $j \notin \{i_1, \dots, i_p\}$ then $v \lrcorner \varphi = 0$ and

$$\begin{aligned} v \wedge \varphi &= e_j \wedge e_{i_1} \wedge \ldots \wedge e_{i_p} \simeq e_j \cdot e_{i_1} \cdot \ldots \cdot e_{i_p} \\ &= v \cdot \varphi \\ &= (-1)^p \varphi \cdot v. \end{aligned}$$

As the equalities of the assertion are bilinear, the proposition is proved. □

2.2. **Classification of Clifford Algebras.**

Now we give some basic assertions which lead to the classification of the complex Clifford algebras. For the real Clifford algebras, we mention without proof the following proposition:

Proposition 2.9. *For all $n \in \mathbb{N}$: $Cl_{n+8} \simeq Cl_n \otimes Cl_8$.*

This fact combined with Proposition 2.7 and the knowledge of Cl_n for $1 \le n \le 8$, yield to the classification of the real Clifford algebras.

Definition 2.10. Let $\mathbb{C}l_n$ denote the *complexification* of the real Clifford algebra Cl_n, i.e. $\mathbb{C}l_n = Cl_n \otimes_{\mathbb{R}} \mathbb{C}$.

Remark 2.11. It is clear that this algebra is isomorphic to the Clifford algebra $Cl(\mathbb{C}^n, g_{\mathbb{C}})$, where $g_{\mathbb{C}}$ is the complex bilinear extension of the standard scalar product on $\mathbb{R}^n$ to $\mathbb{C}^n$ (note that this is a symmetric form, not a Hermitian one).

Proposition 2.12. *The complex Clifford algebras are either isomorphic to $\mathbb{C}(2^m)$, or to $\mathbb{C}(2^m) \oplus \mathbb{C}(2^m)$. In particular:*

$$\mathbb{C}l_{2m} \simeq \mathbb{C}(2^m) \simeq \mathrm{End}_{\mathbb{C}}(\Sigma_{2m})\,, \tag{2.1}$$

$$\mathbb{C}l_{2m+1} \simeq \mathbb{C}(2^m) \oplus \mathbb{C}(2^m) \simeq \mathrm{End}_{\mathbb{C}}(\Sigma_{2m}) \oplus \mathrm{End}_{\mathbb{C}}(\Sigma_{2m}), \tag{2.2}$$

where $\mathbb{C}(2^m)$ denotes the ring of $2^m \times 2^m$ complex matrices, which is an algebra over $\mathbb{C}$, and $\Sigma_{2m} \simeq \mathbb{C}^{2^m}$.

Proof. We only give the proof for $\mathbb{C}l_{2m}$. Let $(e_1, \ldots, e_m, e_{m+1}, \ldots, e_{2m})$ be the canonical orthonormal basis of $\mathbb{R}^{2m}$ and $(z_j, \overline{z}_j)_{j=1,\ldots,m}$ a Witt basis of $\mathbb{R}^{2m} \otimes_{\mathbb{R}} \mathbb{C}$, i.e.

$$z_j := \frac{1}{2}(e_j \otimes 1 - e_{j+m} \otimes i) \quad \text{and} \quad \overline{z}_j := \frac{1}{2}(e_j \otimes 1 + e_{j+m} \otimes i).$$

These vectors satisfy for all $j, k = 1, \ldots, m$ the equations

$$g_{\mathbb{C}}(z_j, z_k) = g_{\mathbb{C}}(\overline{z}_j, \overline{z}_k) = 0\,,$$
$$g_{\mathbb{C}}(\overline{z}_j, z_k) = g_{\mathbb{C}}(z_j, \overline{z}_k) = \frac{1}{2}\delta_{jk}\,,$$

which yield

$$z_j \underset{\mathbb{C}}{\cdot} z_k + z_k \underset{\mathbb{C}}{\cdot} z_j = 0$$
$$\overline{z}_j \underset{\mathbb{C}}{\cdot} \overline{z}_k + \overline{z}_k \underset{\mathbb{C}}{\cdot} \overline{z}_j = 0$$
$$z_j \underset{\mathbb{C}}{\cdot} \overline{z}_k + \overline{z}_k \underset{\mathbb{C}}{\cdot} z_j = -\delta_{jk},$$

since for $x, y \in \mathbb{C}^{2m} \subset \mathbb{C}l_{2m}$: $x \underset{\mathbb{C}}{\cdot} y + y \underset{\mathbb{C}}{\cdot} x = -2g_{\mathbb{C}}(x, y)$. For simplicity we write "$\cdot$" for the complex Clifford multiplication "$\underset{\mathbb{C}}{\cdot}$".

Let $\overline{\omega} = \overline{z}_1 \cdot \ldots \cdot \overline{z}_m$ and observe that $\overline{z}_k \cdot \overline{\omega} = 0$ for $1 \leq k \leq m$ by the formulas above. Denote $z_{L_r} := z_{l_1} \cdot \ldots \cdot z_{l_r}$, for $1 \leq l_1 < \ldots < l_r \leq m$. Then, define

$$\Sigma_{2m} = \mathrm{span}\{\, z_{L_r} \cdot \overline{\omega} \mid 1 \leq l_1 < \ldots < l_r \leq m,\ 0 \leq r \leq m\}.$$

Since Σ_{2m} and $\mathbb{C}^{2^m}$ have the same dimension, they are isomorphic as complex vector spaces.

We define the linear map:

$$\begin{array}{rccl} \rho\colon & \mathbb{C}l_{2m} & \longrightarrow & \mathrm{End}(\Sigma_{2m}) \\ & v = z_{J_p} \cdot \overline{z}_{K_q} & \longmapsto & \rho(v) = (z_{L_r} \cdot \overline{\omega} \ \mapsto z_{J_p} \cdot \overline{z}_{K_q} \cdot z_{L_r} \cdot \overline{\omega}). \end{array}$$

Obviously, ρ is a homomorphism of algebras. We show now that ρ is surjective. First take $v = z_j \cdot \overline{z}_1$ for $1 \leq j \leq m$:

$$\begin{aligned}
\rho(v)\big(z_1 \cdot \overline{\omega}\big) &= z_j \cdot \overline{z}_1 \cdot z_1 \cdot \overline{\omega} \\
&= z_j \cdot (-z_1 \cdot \overline{z}_1 - 1) \cdot \overline{\omega} \\
&= -z_j \cdot z_1 \cdot \underbrace{\overline{z}_1 \cdot \overline{\omega}}_{0} - z_j \cdot \overline{\omega} \\
&= -z_j \cdot \overline{\omega}
\end{aligned}$$

and for $2 \leq l \leq m$

$$\begin{aligned}
\rho(v)\big(z_l \cdot \overline{\omega}\big) &= z_j \cdot \overline{z}_1 \cdot z_l \cdot \overline{\omega} \\
&= z_j \cdot (-z_l \cdot \overline{z}_1) \cdot \overline{\omega} \\
&= 0
\end{aligned}$$

Similarly, one computes that the image under ρ of a basis vector of Σ_{2m} not containing z_1 is zero, whereas the image of a basis vector containing z_1 is the same vector with z_1 being replaced by z_j or $-z_j$.

Denote $z_1 \cdot \ldots \cdot z_m$ by ω. Then, for elements of the Clifford algebra of the form

$$v = z_{J_p} \cdot \overline{\omega} \cdot \omega \cdot \overline{z}_{K_q} := z_{j_1} \cdot \ldots \cdot z_{j_p} \cdot \overline{\omega} \cdot \omega \cdot \overline{z}_{k_1} \cdot \ldots \cdot \overline{z}_{k_q}$$

the map $\rho(v)$ replaces $z_{k_1} \cdot \ldots \cdot z_{k_q}$ by $\pm z_{j_1} \cdot \ldots \cdot z_{j_p}$ in the basis vectors of Σ_{2m}, wheras $\rho(v)$ maps all other basis vectors of Σ_{2m} to zero. Thus ρ is surjective, and since $\dim \mathbb{C}l_{2m} = \dim \mathrm{End}(\mathbb{C}^{2^m})$, it is bijective. $\square$

The above considerations yield to the classification of the complex Clifford algebras. For completeness, we also include the real Clifford algebras in the following table:

n	1	2	3	4	5	6	7	8
Cl_n	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(16)$
$\mathbb{C}l_n$	$\mathbb{C} \oplus \mathbb{C}$	$\mathbb{C}(2)$	$\mathbb{C}(2) \oplus \mathbb{C}(2)$	$\mathbb{C}(4)$	$\mathbb{C}(4) \oplus \mathbb{C}(4)$	$\mathbb{C}(8)$	$\mathbb{C}(8) \oplus \mathbb{C}(8)$	$\mathbb{C}(16)$

2.3. **The Spinor Representation.**

Definition 2.13. In even dimension, the *complex spinor representation*

$$\rho \colon \mathbb{C}l_{2m} \longrightarrow \mathrm{End}_{\mathbb{C}}(\Sigma_{2m})$$

is the isomorphism of Proposition 2.12, whereas in odd dimension, it is defined to be the projection onto the first component of the corresponding isomorphism.

Definition 2.14. The *Clifford multiplication* is the map

$$\begin{array}{rcl} \mathfrak{m}\colon \mathbb{C}l_{2m}\otimes\Sigma_{2m} & \longrightarrow & \Sigma_{2m} \\ \varphi\otimes\sigma & \longmapsto & \varphi\cdot\sigma := \rho(\varphi)(\sigma). \end{array}$$

Proposition 2.15. *Let* $(e_1,\dots,e_n)$ *be a* g*–orthonormal basis of* $\mathbb{R}^n$*. The* complex volume element

$$\omega_{\mathbb{C}} = i^{[\frac{n+1}{2}]}\, e_1\cdot\ldots\cdot e_n$$

of $\mathbb{C}l_n$ *satisfies*
(1) $\omega_{\mathbb{C}}^2 = 1$ *and*
(2) $x\cdot\omega_{\mathbb{C}} = (-1)^{n-1}\omega_{\mathbb{C}}\cdot x$ *for all* $x\in\mathbb{R}^n\subset\mathbb{C}l_n$,
where [ ] *stands for the integer part.*

This yields the following proposition.

Proposition 2.16. *If* $n = 2m$ *is even, the complex spinor representation, restricted to the even part of the Clifford algebra,*

$$\rho^0\colon \mathbb{C}l^0_{2m}\longrightarrow \mathrm{End}_{\mathbb{C}}(\Sigma_{2m})$$

admits a decomposition

$$\Sigma_{2m} = \Sigma^+_{2m}\oplus\Sigma^-_{2m},$$

where

$$\Sigma^{\pm}_{2m} = \frac{1}{2}(1\pm\omega_{\mathbb{C}})\cdot\Sigma_{2m}.$$

Moreover, $\rho^0(x)\colon \Sigma^{\pm}_{2m}\to\Sigma^{\mp}_{2m}$ *is an isomorphism of vector spaces for each* $x\in\mathbb{R}^n\setminus\{0\}$.

2.4. **The Spinor Group.** Denote by Cl_n^* the multiplicative group of units of the real Clifford algebra Cl_n, i.e., the subgroup

$$Cl_n^* = \{\,\varphi\in Cl_n \mid \exists\varphi^{-1}\in Cl_n \text{ such that } \varphi\cdot\varphi^{-1} = \varphi^{-1}\cdot\varphi = 1\,\}.$$

Definition 2.17. The *pin group* of Cl_n is the subgroup Pin_n of Cl_n^*, defined by

$$\mathrm{Pin}_n = \{\, x_1\cdot\ldots\cdot x_k \mid x_j\in\mathbb{R}^n, \|x_j\| = 1\,\}$$

The *spin group* Spin_n then is defined to be

$$\mathrm{Spin}_n = \mathrm{Pin}_n\cap Cl^0_n.$$

In other words, the spin group is the multiplicative subgroup of Cl_n^* generated by even products of vectors of length 1, i.e

$$\mathrm{Spin}_n = \{\, x_1\cdot\ldots\cdot x_{2k} \mid x_j\in\mathbb{R}^n, \|x_j\| = 1\,\}.$$

Remark 2.18.

(1) The inverse of an element $\varphi = x_{i_1} \cdot \ldots \cdot x_{i_{2k}} \in \mathrm{Spin}_n$ is given by ${}^t(\varphi)$.
(2) Denote by $\mathfrak{Cl}_n^*$ the Lie algebra of the Lie group Cl_n^*. Then $\mathfrak{Cl}_n^*$ is isomorphic to Cl_n, the Lie bracket being
$$[\varphi, \psi] = \varphi \cdot \psi - \psi \cdot \varphi$$
for all φ, $\psi \in Cl_n$.

Examples 2.19.

(1) $\mathrm{Spin}_4 \simeq \mathrm{Sp}_1 \times \mathrm{Sp}_1 \simeq \mathrm{SU}_2 \times \mathrm{SU}_2$ where $\mathrm{Sp}_1 = \{ q \in \mathbb{H} \mid \|q\| = 1 \}$. The group Pin_4 is by definition the multiplicative subgroup generated by the elements of Sp_1. The assertion now follows as Cl_4^0 is isomorphic to $\mathbb{H} \oplus \mathbb{H}$ by Proposition 2.7. Furthermore $\mathfrak{spin}_4 \simeq \operatorname{Im} \mathbb{H} \oplus \operatorname{Im} \mathbb{H}$.
(2) Similarly, one sees that $\mathrm{Spin}_3 \simeq \mathrm{Sp}_1 \simeq \mathrm{SU}_2$ and $\mathfrak{spin}_3 \simeq \operatorname{Im} \mathbb{H}$.

The aim of the rest of this section is to show that the spin group Spin_n is the universal 2–fold covering of SO_n, $n \geq 3$. For this purpose, we consider the map
$$\begin{aligned} \mathrm{Ad}_u \colon Cl_n &\longrightarrow Cl_n \\ y &\longmapsto u \cdot y \cdot u^{-1}, \end{aligned}$$
$u \in Cl_n^*$.

Proposition 2.20. *For all $x \in \mathbb{R}^n$, $\|x\| = 1$, the map Ad_x is an endomorphism of $\mathbb{R}^n$. Furthermore, $-\mathrm{Ad}_x$ is the reflection across $x^\perp$.*

Proof. For $x \in \mathbb{R}^n$, $\|x\| = 1$, we have $x^{-1} = -x$ in Cl_n. Thus, for $y \in \mathbb{R}^n$,
$$-\mathrm{Ad}_x(y) = x \cdot y \cdot x = x \cdot (-x \cdot y - 2g(x, y)) = y - 2g(x, y)x$$
□

For $u = x_1 \ldots x_{2k} \in \mathrm{Spin}_n$ and $y \in \mathbb{R}^n$, we get
$$\begin{aligned} \mathrm{Ad}_u(y) &= u \cdot y \cdot u^{-1} \\ &= u \cdot y \cdot {}^t u \\ &= x_1 \cdot \ldots \cdot x_{2k} \cdot y \cdot x_{2k} \cdot \ldots \cdot x_1 \\ &= \mathrm{Ad}_{x_1} \circ \ldots \circ \mathrm{Ad}_{x_{2k}}. \end{aligned}$$
This is an even composition of reflections of $\mathbb{R}^n$, hence an element of SO_n. By the Cartan-Dieudonné theorem, every element of SO_n is a product of an even number of reflections. We thus have the proposition:

Proposition 2.21. *The linear map $\mathrm{Ad}\big|_{\mathrm{Spin}_n} \colon \mathrm{Spin}_n \longrightarrow \mathrm{SO}_n$ is surjective.*

$\mathrm{Ad}\big|_{\mathrm{Spin}_n}$ is not injective, but we have:

Proposition 2.22. *The sequence*

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}_n \xrightarrow{\mathrm{Ad}} \mathrm{SO}_n \longrightarrow 1$$

is a short exact sequence. Furthermore, if $n \geq 3$, Spin_n *is the universal covering of* SO_n.

Proof. An element $u \in \mathrm{Spin}_n \subset Cl_n^0$ can be decomposed into $u = a_0 + e_1 \cdot a_1$, such that $a_0 \in Cl_n^0$ and $a_1 \in Cl_n^1$, a_0 and a_1 not containing e_1. Thus $a_0 \cdot e_1 = e_1 \cdot a_0$ and $e_1 \cdot a_1 \cdot e_1 = -e_1 \cdot e_1 \cdot a_1 = a_1$. Suppose now that u is in the kernel of Ad, i.e., for all $y \in \mathbb{R}^n$ $\mathrm{Ad}_u(y) = y \Leftrightarrow u \cdot y = y \cdot u$. For $y = e_1$, we then get:

$$\begin{aligned}(a_0 + e_1 \cdot a_1) \cdot e_1 = e_1 \cdot (a_0 + e_1 \cdot a_1) &\implies a_0 \cdot e_1 + e_1 \cdot a_1 \cdot e_1 = e_1 \cdot a_0 - a_1 \\ &\implies a_1 = -a_1 \implies a_1 = 0.\end{aligned}$$

Hence u does not contain e_1. As the same procedure works for all the e_j's, we get $u \in \{-1, 1\}$ and $\ker \mathrm{Ad} = \{-1, 1\}$.

To prove that the covering is non trivial for $n \geq 3$, it suffices to find a continuous path in Spin_n which joints -1 to the 1. One can easily see this using the path:

$$\begin{aligned}\gamma(t) &= \big(\cos(\frac{t}{2})e_i + sin(\frac{t}{2})e_j\big) \cdot \big(-\cos(\frac{t}{2})e_i + sin(\frac{t}{2})e_j\big) \\ &= \cos(t) + \sin(t) e_i \cdot e_j. \end{aligned} \tag{2.3}$$

□

Proposition 2.23. *The homomorphism* $\mathrm{Ad}_* \colon \mathfrak{spin}_n \to \mathfrak{so}_n (\simeq \Lambda^2 \mathbb{R}^n)$ *between the Lie algebras associated to* Spin_n *and* SO_n *is a vector space isomorphism. It is given by*

$$\Big(\mathrm{Ad}_*(e_i \cdot e_j)\Big)(y) = 2\big(e_i \wedge e_j\big)(y) = 2g(e_i, y)e_j - 2g(e_j, y)e_i$$

for $1 \leq i, j \leq n$.

Proof. Consider the path $\gamma(t)$ defined in (2.3). As $\gamma(0) = 1$, $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\gamma(t) = e_i \cdot e_j$ and $\mathfrak{spin}_n$ is isomorphic to $\mathrm{T}_1 \mathrm{Spin}_n$, we can assume that $e_i \cdot e_j$ lies in $\mathfrak{spin}_n$, $1 \leq i, j \leq n$, $i \neq j$.

We have:

$$\begin{aligned}\Big(\mathrm{Ad}_*(e_i \cdot e_j)\Big)(y) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\big(\mathrm{Ad}_{\gamma(t)}(y)\big) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\big(\gamma(t) \cdot y \cdot \gamma^{-1}(t)\big) \\ &= \gamma'(0) \cdot y \cdot \gamma(0) + \gamma(0) \cdot y \cdot (\gamma^{-1})'(0) \\ &= e_i \cdot e_j \cdot y - y \cdot e_i \cdot e_j \\ &= e_i \cdot e_j \cdot y - \big(-e_i \cdot y - 2g(e_i, y)\big) \cdot e_j \\ &= e_i \cdot e_j \cdot y + e_i \cdot \big(-e_j \cdot y - 2g(y, e_j)\big) + 2g(e_i, y)e_j \\ &= 2\big(e_i \wedge e_j\big)(y)\end{aligned}$$

which is the stated formula and additionally proves that Ad_* is surjective. Since the dimensions of both, $\mathfrak{spin}_n$ and $\Lambda^2\mathbb{R}^n$, are equal, Ad_* is an isomorphism. □

We end by emphasizing that the spin group Spin_n is a compact, connected, simply connected (for $n \geq 3$) Lie group of dimension $\frac{n(n-1)}{2}$.

2.5. **The Space of Complex Spinors.** For simplicity, we restrict to the complex case.

Definition 2.24. Let $\rho\colon \mathbb{C}l_n \to \mathrm{End}(\Sigma_n)$ be an irreducible representation of $\mathbb{C}l_n$.Then the restriction of ρ to Spin_n

$$\rho\colon \mathrm{Spin}_n \longrightarrow \mathrm{Aut}(\Sigma_n).$$

is called the *complex spinor representation* and Σ_n the *space of complex spinors*, $\dim_{\mathbb{C}}(\Sigma_n) = 2^{[\frac{n}{2}]}$.

Proposition 2.16 can now be formulated as:

Theorem 2.25. *If $n = 2m$ is even, the complex spinor representation of* Spin_{2m} *decomposes into*

$$\rho = \rho^+ + \rho^-.$$

That is, the space of spinors decomposes into posive and negative spinors, $\Sigma_{2m} = \Sigma^+_{2m} \oplus \Sigma^-_{2m}$, where $\Sigma^\pm_{2m} = \frac{1}{2}(1 \pm \omega_{\mathbb{C}}) \cdot \Sigma_{2m}$, so that

$$\rho^\pm\colon \mathrm{Spin}_{2m} \longrightarrow \mathrm{Aut}(\Sigma^\pm_{2m}).$$

Furthermore, for $x \in \mathbb{R}^n \setminus \{0\}$, the map

$$\begin{array}{rccc} x\cdot : & \Sigma^\pm_{2m} & \longrightarrow & \Sigma^\mp_{2m} \\ & \sigma & \longmapsto & x\cdot\sigma\,, \end{array}$$

is an isomorphism. The maps $\rho^\pm$ are inequivalent, irreducible, complex representations of Spin_{2m}.

For $n = 2m+1$ odd, the spinor representation is irreducible and does not depend on the projection on the components of $\mathrm{End}(\Sigma_{2m}) \oplus \mathrm{End}(\Sigma_{2m})$ *chosen in (2.14).*

Proposition 2.26 (The natural Hermitian product)**.** *There exists on Σ_n, a* natural Hermitian scalar product *such that*

$$(\sigma_1, \sigma_2) = (x\cdot\sigma_1, x\cdot\sigma_2)$$

for every $x \in \mathbb{R}^n$, $\|x\| = 1$, and σ_1, $\sigma_2 \in \Sigma_n$.

Proof. Let Γ_n be the multiplicative subgroup of Cl^*_n generated by a g–orthonormal basis $\{e_1, \dots, e_n\}$ of $\mathbb{R}^n$. Using the relations $(-1)^2 = 1$, $e_i^2 = -1$ and $e_i \cdot e_j = -e_j \cdot e_i$, $1 \leq i, j \leq n$, $i \neq j$, we see that Γ_n is finite and $|\Gamma_n| = 2^{n+1}$. Now we choose an arbitrary Hermitian product $\langle \cdot, \cdot \rangle$ on Σ and define for $\sigma_1,\ \sigma_2 \in \Sigma_n$

$$(\sigma_1, \sigma_2) = \frac{1}{|\Gamma_n|} \sum_{\nu \in \Gamma_n} \langle \nu \cdot \sigma_1, \nu \cdot \sigma_2 \rangle.$$

First, for $e_i \in \Gamma_n$, it follows

$$\begin{aligned}
(e_i \cdot \sigma_1, e_i \cdot \sigma_2) &= \frac{1}{|\Gamma_n|} \sum_{\nu \in \Gamma_n} \langle \nu \cdot e_i \cdot \sigma_1, \nu \cdot e_i \cdot \sigma_2 \rangle \\
&= \frac{1}{|\Gamma_n|} \sum_{\nu \in \Gamma_n} \langle \nu \cdot \sigma_1, \nu \cdot \sigma_2 \rangle \\
&= (\sigma_1, \sigma_2)\,.
\end{aligned}$$

Then, for $x \in \mathbb{R}^n$ with $\|x\| = 1$, i.e., $x = \sum_i^n x_i e_i$, with $\sum_i x_i^2 = 1$, we get

$$\begin{aligned}
(x \cdot \sigma_1, x \cdot \sigma_2) &= \sum_i x_i^2 (e_i \cdot \sigma_1, e_i \cdot \sigma_2) + \sum_{i \neq j} x_i x_j (e_i \cdot \sigma_1, e_j \cdot \sigma_2) \\
&= \sum_i x_i^2 (\sigma_1, \sigma_2) \\
&\quad + \sum_{i<j} x_i x_j \Big((e_i \cdot \sigma_1, e_j \cdot \sigma_2) + (e_j \cdot \sigma_1, e_i \cdot \sigma_2) \Big) \\
&= (\sigma_1, \sigma_2)
\end{aligned}$$

since, for $i < j$, one has

$$\begin{aligned}
(e_i \cdot \sigma_1, e_j \cdot \sigma_2) &= (e_i \cdot e_i \cdot \sigma_1, e_i \cdot e_j \cdot \sigma_2) \\
&= -(\sigma_1, e_i \cdot e_j \cdot \sigma_2) \\
&= -(e_j \cdot \sigma_1, e_j \cdot e_i \cdot e_j \cdot \sigma_2) \\
&= (e_j \cdot \sigma_1, e_i \cdot e_j \cdot e_j \cdot \sigma_2) \\
&= -(e_j \cdot \sigma_1, e_i \cdot \sigma_2)
\end{aligned}$$

□

Corollary 2.27. *For all $x \in \mathbb{R}^n$ and for all $\sigma_1,\ \sigma_2 \in \Sigma_n$, we have*

$$(x \cdot \sigma_1, \sigma_2) = -(\sigma_1, x \cdot \sigma_2).$$

Proof. Let x be in $\mathbb{R}^n \setminus \{0\}$. Then $(x \cdot \sigma_1, \sigma_2) = (x \cdot \frac{x}{\|x\|} \cdot \sigma_1, \frac{x}{\|x\|} \cdot \sigma_2)$ and

$$(x \cdot \sigma_1, \sigma_2) = \frac{1}{\|x\|^2} (x \cdot x \cdot \sigma_1, x \cdot \sigma_2) = -(\sigma_1, x \cdot \sigma_2).$$

□

3. Connections on Vector and Principal Fibre Bundles

For convenience, we introduce the notion of vector and principal fibre bundles [KN] and their relationship. We then define the notion of connection form and the associated notion of covariant derivative. With this, we canonically define the covariant derivative on the spin bundle in terms of the Levi-Civita connection on the tangent bundle.

3.1. Vector and Principal Fibre Bundles.

Definition 3.1. A *vector bundle of rank* N over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is a triple $(E,\, \pi,\, M)$ such that

i) the *projection* $\pi\colon E \to M$ is a smooth map between smooth manifolds,
ii) for all $x \in M$, the *fibre at* x, $E_x := \pi^{-1}(x)$, is an N–dimensional vector space over $\mathbb{K}$,
iii) for every $x \in M$, there exists an open neighbourhood $U \subset M$ and a diffeomorphism, called *local trivialisation*,
$$\phi\colon \pi^{-1}(U) \to U \times \mathbb{K}^N$$
such that for all $y \in U$,
$$\phi|_{E_y}\colon E_y \to \{y\} \times \mathbb{K}^N$$
is an isomorphism of vector spaces.

Examples 3.2.

1) The *trivial bundle* $M \times \mathbb{R}^N$.
2) The *tangent bundle* TM of a smooth manifold M.

Proposition 3.3 (Transition functions)**.** *Let* $(E,\, \pi,\, M)$ *be a vector bundle and* $(U_\alpha,\, \phi_\alpha)_{\alpha\in A}$ *a cover of local trivialisations. Then the* transition functions
$$\varphi_{\beta\alpha}\colon U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}(N, \mathbb{K}) =: \mathrm{GL}_N,$$
defined by
$$\begin{array}{rccc} \phi_\beta \circ \phi_\alpha^{-1}\colon & \big(U_\alpha \cap U_\beta\big) \times \mathbb{K}^N & \longrightarrow & \big(U_\alpha \cap U_\beta\big) \times \mathbb{K}^N \\ & (x\,,\, v) & \longmapsto & \big(x\,,\, \varphi_{\beta\alpha}(x)v\big), \end{array}$$
satisfy the cocycle condition
$$\varphi_{\gamma\beta} \circ \varphi_{\beta\alpha} = \varphi_{\gamma\alpha}\colon U_\alpha \cap U_\beta \cap U_\gamma \to \mathrm{GL}_N$$
for all $\alpha,\, \beta,\, \gamma \in A$ *and all* $x \in U_\alpha \cap U_\beta \cap U_\gamma$.

Proposition 3.4. *Let* $(U_\alpha)_{\alpha\in A}$ *be a cover of* M *and let*
$$\varphi_{\beta\alpha}\colon U_\alpha \cap U_\beta \to \mathrm{GL}_N$$
be differentiable maps satisfying
$$\varphi_{\gamma\alpha} = \varphi_{\gamma\beta} \circ \varphi_{\beta\alpha}\colon U_\alpha \cap U_\beta \cap U_\gamma \to \mathrm{GL}_N\,.$$

If we define

$$E := \Big(\coprod_{\alpha\in A} U_\alpha \times \mathbb{K}^N\Big)\Big/\sim$$

where

$$(x_\alpha, v) \sim (x_\beta, w) :\Longleftrightarrow x = x_\alpha = x_\beta \in U_\alpha \cap U_\beta \quad \textit{and} \quad w = \varphi_{\beta\alpha}(x)v\,,$$

then E defines a rank N vector bundle over $\mathbb{K}$.

Definition 3.5. Given a Lie group G, a *G–principal fibre bundle* is a triple $(P,\, \pi,\, M)$ such that

i) $\pi\colon P \to M$ is a smooth map between smooth manifolds,

ii) G acts freely on P from the right, i.e. the action $P \times G \to P$ satisfies $p\,g = p$ iff $g = e \in G$,

iii) for every point $x \in M$, ther exists an open neighbourhood $U \subset M$ and a diffeomorphism, called *local trivialisation*,

$$\phi\colon \pi^{-1}(U) \to U \times G, \quad p \mapsto (\pi(p),\, \varphi(p))\,,$$

such that

$$\phi(p\,g) = (\pi(p\,g),\, \varphi(p\,g)) = (\pi(p),\, \varphi(p)\,g)$$

for all $p \in U$ and all $g \in G$. In particular, $\phi|_{\pi^{-1}(p)}$ is a diffeomorphism of the fibre $P_p := \pi^{-1}(p)$ onto $\{p\} \times G$ which commutes with the action of G on P.

For a cover $(U_\alpha,\, \phi_\alpha)$ of local trivialisations, we define as above the transition functions $\varphi_{\beta\alpha}\colon U_\alpha \cap U_\beta \to G$ by

$$\begin{array}{rcl} \phi_\beta \circ \phi_\alpha^{-1}\colon \big(U_\alpha \cap U_\beta\big) \times G & \longrightarrow & \big(U_\alpha \cap U_\beta\big) \times G \\ (x,\, g) & \longmapsto & (x,\, \varphi_{\beta\alpha}(x)\, g). \end{array}$$

and again get the cocycle conditions $\varphi_{\gamma\beta}\circ\varphi_{\beta\alpha} = \varphi_{\gamma\alpha}$. As in Proposition 3.4, one can reconstruct the principal fibre bundle out of the transition functions.

Examples 3.6. 1) Every smooth manifold M^n comes naturally with the GL_n–principal fibre bundle $\mathrm{GL}\,M \xrightarrow{\pi} M$ with typical fibre at $x \in M$

$$(\mathrm{GL}\,M)_x = \{\text{ordered bases of } T_xM\}\,.$$

2) The natural fibre bundle of an oriented Riemannian manifold (M^n, g) is the SO_n–principal fibre bundle $\mathrm{SO}\,M$ of oriented orthonormal bases at $x \in M$, i.e.,

$$(\mathrm{SO}\,M)_x = \{\text{positive oriented orthonormal bases of } T_xM\}.$$

Let $(P,\, \pi,\, M)$ be a G–principal fibre bundle. We take a finite dimensional representation

$$\rho\colon G \longrightarrow \mathrm{End}(\Sigma)$$

of G on a vector space Σ and define an action of G on $P\times\Sigma$ as follows:

$$\begin{array}{rcl}(P\times\Sigma)\times G & \longrightarrow & P\times\Sigma\\ (p,v,g) & \longmapsto & (p\,g,\rho(g^{-1})v).\end{array}$$

Dividing $P\times\Sigma$ by the equivalence relation $(p,v)\sim(p\,g,\rho(g^{-1})v)$ gives the *associated vector bundle*

$$E := P\times_\rho\Sigma = (P\times\Sigma)/\sim\, = (P\times\Sigma)/G.$$

The transition functions of E are $\rho\circ\varphi_{\beta\alpha}$, where $\varphi_{\beta\alpha}$ are the transition functions of P.

Example 3.7. In case of the trivial G–principal fibre bundle $P=M\times G$, we get $(M\times G)\times_\rho\Sigma\simeq M\times\Sigma$ for any representation ρ of G on Σ.

The tangent bundle of an oriented Riemannian manifold can be understood as $TM\simeq \mathrm{GL}\,M\times_{\rho_0}\mathbb{R}^n\simeq \mathrm{SO}\,M\times_{\rho_0}\mathbb{R}^n$, where ρ_0 stands for the standard matrix representation of GL_n and SO_n on $\mathbb{R}^n$.

On the other hand, let $(E,\,\pi,\,M)$ be a vector bundle of rank N with structure group G, i.e. $G\subset\mathrm{GL}_N$ is a closed subgroup. If the transition functions $\varphi_{\beta\alpha}$ of a cover of local trivialisations $(U_\alpha,\,\phi_\alpha)$ can be chosen to have values in G, this defines a G–principal fibre bundle P, to which E is associated. We obtain a representation of G by composing the inclusion map of G into GL_N and the canonical representation of GL_N on $\mathbb{K}^N$.

3.2. **The Connection Form and the Covariant Derivative.**

Definition 3.8. Let $(E,\,\pi,\,M)$ be a vector bundle. A *covariant derivative* is a linear map

$$\nabla\colon\Gamma(E)\longrightarrow\Gamma(T^*M\otimes E),$$

which satisfies the *Leibniz rule*

$$\nabla(f\psi)=\mathrm{d}f\otimes\psi+f\ \nabla\psi.$$

Remark 3.9. If we take a vector field $X\in\Gamma(TM)$ and evaluate $\nabla_X\psi$ at $x\in M$, then $\big(\nabla_X\psi\big)(x)$ depends only on the vector X_x and the values of ψ in an arbitrary small neighbourhood of x.

Definition 3.10. Let $(P,\,\pi,\,M)$ be a G–principal fibre bundle. For any point $p\in P$, there is a canonical injection

$$\begin{array}{rcl}\sim\colon\mathfrak{g} & \longrightarrow & T_pP\\ z & \longmapsto & \widetilde{z}_p=\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}(p\ \exp(tz)).\end{array}$$

Its image is called the *vertical space* V_p and is the tangent space to the fibre $\pi^{-1}(p)$, i.e. $V_p=\ker(\pi_*)$.

Definition 3.11. A *connection* in a G–principal fibre bundle $(P,\, \pi,\, M)$ is a distribution of n-dimensional vector spaces $p \mapsto H_p \subset T_pP$, the *horizontal spaces*, such that

i) $T_pP = V_p \oplus H_p$, and
ii) it is G–invariant, i.e. $H_{pg} = \big(\mathrm{R}_g\big)_*(H_p)$, where $\mathrm{R}_g \colon P \to P$, $p \mapsto pg$.

The projection π induces an isomorphism $\pi_*|_{H_p} \colon H_p \to T_{\pi(p)}M$.

Proposition 3.12. *Let $(P,\, \pi,\, M)$ be a G-principal fibre bundle. The decomposition of T_pP by a connection allows us to define a 1–form $\boldsymbol{\omega}$ on TP with values in the Lie algebra $\mathfrak{g}$ of G*

$$\boldsymbol{\omega}_p \colon T_pP = V_p \oplus H_p \xrightarrow{proj} V_p \xrightarrow{\sim^{-1}} \mathfrak{g},$$

that has the following properties

i) $\boldsymbol{\omega}_p(\widetilde{z}) = z$, *where $\widetilde{z}$ is defined in (3.10),*
ii) $\mathrm{R}_g^*\boldsymbol{\omega} = \mathrm{ad}(g^{-1})\boldsymbol{\omega}$, *i.e.,*

$$\forall X \in \Gamma(TP), \quad \boldsymbol{\omega}\big((\mathrm{R}_g)_*X\big) = \mathrm{ad}(g^{-1})\, \boldsymbol{\omega}(X)\,,$$

where $\mathrm{ad}\colon G \longrightarrow \mathrm{End}(\mathfrak{g})$, $g \mapsto \mathrm{d}\alpha_g$ *and* $\alpha_g \colon G \longrightarrow G$, $a \mapsto gag^{-1}$.

Conversely, a 1–form on TP with values in $\mathfrak{g}$ which satisfies i) and ii) defines a connection on P by $H_p := \ker \boldsymbol{\omega}_p$.

For a connection 1–form $\boldsymbol{\omega}$ on a G–principal fibre bundle $(P,\, \pi,\, M)$, we define a covariant derivative on every associated vector bundle $E = P \times_\rho \Sigma$ as follows:

Take a section $\psi \in \Gamma(E)$, which is locally given by $\psi = [s,\, \sigma]$, where $s \in \Gamma_U(P)$ is a local section on $U \subset M$ and $\sigma \colon U \to \Sigma$, a function. Since

$$TU \xrightarrow{s_*} TP \xrightarrow{\boldsymbol{\omega}} \mathfrak{g} \xrightarrow{\rho_*} \mathrm{End}(\Sigma),$$

we can define a covariant derivative on E by

$$\nabla_X\psi := [\, s,\, X\sigma + \rho_*((\boldsymbol{\omega} \circ s_*)(X))\sigma\,] \tag{3.1}$$

for any $X \in TU$, where $X\sigma$ denotes the Lie derivative of σ in the direction of X.

Conversely, given a vector bundle $(E,\, \pi,\, M)$ of rank N and a covariant derivative ∇, we consider N lineary independent local sections of E

$$s = (\psi_1, \ldots, \psi_N) \colon U \to \mathrm{GL}\, E, \quad U \subset M$$

and define 1–forms $\omega_{\beta\alpha}$ by

$$\nabla_X\psi_\alpha =: \sum_{\beta=1}^{N} \omega_{\beta\alpha}(X)\psi_\beta \tag{3.2}$$

for all $X \in \Gamma(TU)$.

Now, there is a unique connection form $\boldsymbol{\omega}$ on the GL_N–principal fibre bundle $\mathrm{GL}\,E$ such that for any local section $s \in \Gamma_U(\mathrm{GL}\,E)$

$$s^*\boldsymbol{\omega} := \omega = (\omega_{\beta\alpha})_{1\le\alpha,\beta\le N} \in \mathfrak{g} = \mathrm{End}_N\,.$$

Notice that, for any local tangent vector field X, one has to define

$$\boldsymbol{\omega}(s_*X + \widetilde{z}) := \big(\omega_{\beta\alpha}(X)\big) + z\,.$$

.

Remark 3.13. If the vector bundle E is endowed with a metric g and a metric connection ∇, i.e.,

$$Xg(\psi,\varphi) = g(\nabla_X\psi,\varphi) + g(\psi,\nabla_X\varphi)\,, \quad \forall\psi,\varphi \in \Gamma(E), \quad X \in \Gamma(TM)\,,$$

then, the corresponding matrix of 1–forms $(\omega_{\beta\alpha})$ is skew symmetric with respect to an arbitrary orthonormal frame $s = (\psi_1,\ldots,\psi_N)\colon U \to \mathrm{SO}\,E$. It is therefore an element of the Lie algebra $\mathfrak{so}_N$ of SO_N. To see this, take $X \in \Gamma(TM)$ to get

$$g(\nabla_X\psi_\alpha,\psi_\beta) = -g(\psi_\alpha,\nabla_X\psi_\beta)$$

for the metric connection ∇. Thus

$$\begin{aligned}\omega_{\beta\alpha}(X) &= g(\sum_\gamma \omega_{\gamma\alpha}(X)\,\psi_\gamma,\psi_\beta)\\ &= g(\nabla_X\psi_\alpha,\psi_\beta)\\ &= -g(\psi_\alpha,\nabla_X\psi_\beta)\\ &= -\omega_{\alpha\beta}(X).\end{aligned}$$

3.3. **The Curvature.** For a G–principal fibre bundle $(P,\,\pi,\,M)$ with connection form $\boldsymbol{\omega}$, define the *curvature 2–form* $\boldsymbol{\Omega}$ by:

$$\begin{gathered}\boldsymbol{\Omega} \in \Gamma(\Lambda^2(TP)\otimes\mathfrak{g})\\ \boldsymbol{\Omega}(X,Y) = \mathrm{d}\boldsymbol{\omega}(X,Y) + [\boldsymbol{\omega}(X),\boldsymbol{\omega}(Y)], \quad X,Y \in \Gamma(TP)\end{gathered}$$

As in (3.13), with respect to a local section $s = (\psi_1,\ldots,\psi_N)\colon U \to \mathrm{SO}\,E$, $U \subset M$, let $\Omega = s^*\boldsymbol{\Omega}$, then one gets the following relation

$$\Omega_{\alpha\beta} = (\mathrm{d}\omega)_{\alpha\beta} + \sum_{\gamma=1}^{N}\omega_{\alpha\gamma}\wedge\omega_{\gamma\beta}.$$

Definition 3.14. Let $(E,\,\pi,\,M)$ be a vector bundle with a metric connection ∇. Define the *curvature tensor*

$$\mathcal{R}\colon \Gamma(E) \xrightarrow{\nabla} \Gamma(T^*M\otimes E) \xrightarrow{\widetilde{\nabla}} \Gamma(\Lambda^2(T^*M)\otimes E),$$

where

$$\widetilde{\nabla}(\alpha\otimes\psi) := \mathrm{d}\alpha\otimes\psi - \alpha\wedge\nabla\psi\,. \tag{3.3}$$

Proposition 3.15. *For a local section $s = (\psi_1, \ldots, \psi_N) \in \Gamma_U(\mathrm{SO}\, E)$ we have*

$$\mathcal{R}\,\psi_\alpha = \sum_{\beta=1}^{N} \Omega_{\beta\alpha} \otimes \psi_\beta.$$

Proof. By defintion of $\widetilde{\nabla}$ and (3.2), it follows

$$\begin{aligned}
\widetilde{\nabla}(\nabla(\psi_\alpha)) &= \widetilde{\nabla}\Big(\sum_\beta \omega_{\beta\alpha} \otimes \psi_\beta\Big) \\
&= \sum_\beta \left(\mathrm{d}\omega_{\beta\alpha} \otimes \psi_\beta - \omega_{\beta\alpha} \wedge \Big(\sum_\gamma \omega_{\gamma\beta} \otimes \psi_\gamma\Big)\right) \\
&= \sum_\beta \left(\mathrm{d}\omega_{\beta\alpha} - \sum_\gamma \omega_{\gamma\alpha} \wedge \omega_{\beta\gamma}\right) \otimes \psi_\beta \\
&= \sum_\beta \left(\mathrm{d}\omega_{\beta\alpha} + \sum_\gamma \omega_{\beta\gamma} \wedge \omega_{\gamma\alpha}\right) \otimes \psi_\beta \\
&= \sum_\beta \Omega_{\beta\alpha} \otimes \psi_\beta.
\end{aligned}$$

□

Proposition 3.16. *Definition 3.14 coincides with the usual one, i.e.,*

$$\mathcal{R}_{X,Y} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

Proof. Take local sections X, $Y \in \Gamma(TM)$, and $\psi \in \Gamma_U(E)$, then there are $\alpha_i \in \Gamma_U(T^*M)$, $i = 1, \ldots, n$, and $\psi_\beta \in \Gamma_U(E)$, $\beta = 1, \ldots, N$ such that

$$\nabla\psi = \sum_{i,\beta} \alpha_i \otimes \psi_\beta\,.$$

Using (3.2) and (3.3), one gets

$$\begin{aligned}
\widetilde{\nabla}(\nabla\psi)(X,Y) &= \sum_{i,\beta} (\mathrm{d}\alpha_i \otimes \psi_\beta)(X,Y) - (\alpha_i \wedge \nabla\psi_\beta)(X,Y) \\
&= \sum_{i,\beta} (X\alpha_i(Y) - Y\alpha_i(X) - \alpha_i([X,Y]))\,\psi_\beta \\
&\quad - \alpha_i(X)\nabla_Y\psi_\beta + \alpha_i(Y)\nabla_X\psi_\beta \\
&= \sum_{i,\beta} \nabla_X(\alpha_i(Y)\psi_\beta) - \nabla_Y(\alpha_i(X)\psi_\beta) - \alpha_i([X,Y])\psi_\beta \\
&= \big(\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}\big)\psi.
\end{aligned}$$

□

4. Spin Structures and the Dirac Operator

Here we introduce the notion of spin structure which is needed to globally define the spinor bundle. We then extend, to this bundle, the algebraic properties introduced in Section 2 and compute the local expression of the spinorial covariant derivative as well as the spinorial curvature. We then end with the definition of the Dirac operator, its basic properties and give a proof of the Schrödinger-Lichnerowicz formula.

4.1. **The Spinor Bundle.**

Definition 4.1. Let $(M^n,\, g)$ be an n-dimensional Riemannian manifold. A *spin structure* on M is a pair $(\operatorname{Spin} M,\, \eta)$, where $\operatorname{Spin} M$ is a Spin_n-principal fibre bundle over M and η a 2–fold covering such that the following diagram commutes:

$$\begin{tikzcd}
\operatorname{Spin} M \times \operatorname{Spin}_n \arrow[r] \arrow[d, "\eta \times \mathrm{Ad}"] & \operatorname{Spin} M \arrow[d, "\eta"] \arrow[dr, "\pi"] & \\
\mathrm{SO}\, M \times \mathrm{SO}_n \arrow[r] & \mathrm{SO}\, M \arrow[r, "\pi"'] & M
\end{tikzcd}$$

The maps in the rows are respectively, the actions of Spin_n and SO_n on the principal fibre bundles $\operatorname{Spin} M$ and $\mathrm{SO}\, M$.

The existence of a spin structure on M is equivalent to the fact that, for the transition functions $\varphi_{\beta\alpha}$ of $\mathrm{SO}\, M$, there is a choice of lifts to transition functions of $\operatorname{Spin} M$, i.e., the diagram

$$\begin{tikzcd}
 & \operatorname{Spin}_n \arrow[d, "\mathrm{Ad}"] \\
M \supset U_\alpha \cap U_\beta \arrow[ur, "\widetilde{\varphi}_{\beta\alpha}"] \arrow[r, "\varphi_{\beta\alpha}"] & \mathrm{SO}_n
\end{tikzcd}$$

commutes and the $\widetilde{\varphi}_{\beta\alpha}$ satisfy the cocycle conditions

$$\widetilde{\varphi}_{\gamma\beta}\widetilde{\varphi}_{\beta\alpha} = \widetilde{\varphi}_{\gamma\alpha}.$$

This, anyhow, is equivalent to the second Stiefel–Whitney class $\omega_2(M)$ being zero. For example, in even complex dimensions, the complex projective space is not spin (see [LM] for a careful treatment of such questions).

Definition 4.2. i) The *complex spinor bundle* associated to a spin structure $\operatorname{Spin} M$ of M is the complex vector bundle

$$\Sigma M := \operatorname{Spin} M \times_\rho \Sigma_n,$$

where $\rho\colon \operatorname{Spin}_n \to \operatorname{Aut}(\Sigma_n)$ is the complex Spin_n representation, $\Sigma_n \simeq \mathbb{C}^N$ and $N = 2^{[\frac{n}{2}]}$ (see Definition 2.24). A section $\psi \in$

$\Gamma(\Sigma M)$ is locally given by

$$\psi|_U = [\widetilde{s}, \sigma],$$

where $\widetilde{s} \in \Gamma_U(\operatorname{Spin} M)$, $U \subset M$ and $\sigma\colon U \to \Sigma_n$.

ii) The *Clifford multiplication* on ΣM is the fiberwise action given by

$$\begin{array}{rccl} \mathfrak{m}\colon & TM \otimes \Sigma M & \longrightarrow & \Sigma M \\ & X \otimes \psi = [\widetilde{s}, \alpha] \otimes [\widetilde{s}, \sigma] & \longmapsto & [\widetilde{s}, \alpha \cdot \sigma] =: X \cdot \psi, \end{array}$$

where $\alpha \cdot \sigma$ is the Clifford multiplication on Σ_n (defined in (2.14)), where the tangent bundle TM is seen as the associated vector bundle $TM \simeq \operatorname{Spin} M \times_{\mathrm{Ad}} \mathbb{R}^n$.

iii) The *natural Hermitian product* on ΣM is defined by

$$\begin{array}{rcl} (\cdot,\cdot) : \Gamma(\Sigma M) \times \Gamma(\Sigma M) & \longrightarrow & C^\infty(M, \mathbb{C}) \\ \psi \otimes \varphi & \longmapsto & (\psi, \varphi), \end{array}$$

where for all $x \in M$, $(\psi, \varphi)_x := (\psi_x, \varphi_x)$ is the natural Hermitian product on Σ_n (see Proposition 2.26).

With the help of Corollary 2.27, it is straightforward to check that for all $X \in \Gamma(TM)$, ψ, $\varphi \in \Gamma(\Sigma M)$, the Clifford multiplication and the Hermitian product on ΣM satisfy the compatibility condition

$$(X \cdot \psi, \varphi) + (\psi, X \cdot \varphi) = 0\,. \tag{4.1}$$

4.2. **The Spinorial Levi–Civita Connection.** Take a simply connected open subset $U \subset M$. Then any local section $s \in \Gamma_U(\mathrm{SO}\, M)$ lifts to a section $\widetilde{s} \in \Gamma_U(\operatorname{Spin} M)$, i.e.,

$$\begin{tikzcd} & \operatorname{Spin} M \arrow[d, "\eta"] \\ U \subset M \arrow[ur, "\widetilde{s}"] \arrow[r, "s"'] & \mathrm{SO}\, M \end{tikzcd}$$

and we can define a connection 1–form $\widetilde{\boldsymbol{\omega}}$ on $\operatorname{Spin} M$ as the unique connection 1–form for which the following diagram commutes

$$\begin{tikzcd} & \mathrm{T}\operatorname{Spin} M \arrow[r, "\widetilde{\boldsymbol{\omega}}"] \arrow[d, "\eta_*"] & \mathfrak{spin}_n \arrow[d, "\mathrm{Ad}_*"] \\ TU \subset TM \arrow[ur, "\widetilde{s}_*"] \arrow[r, "s_*"'] & \mathrm{T}\,\mathrm{SO}\, M \arrow[r, "\boldsymbol{\omega}"'] & \mathfrak{so}_n \end{tikzcd} \tag{4.2}$$

The connection 1–form $\boldsymbol{\omega}$ is given by the Levi–Civita connection on (M, g).

To get a local description of the associated covariant derivative ∇ on ΣM, take an orthonormal frame $s = (e_1, \dots, e_n) \in \Gamma_U(\mathrm{SO}\, M)$, $U \subset M$, and denote by:

$$\omega := s^* \boldsymbol{\omega} = -\sum_{i<j} \omega_{ij}\, e_i \wedge e_j$$
$$\Omega := s^* \boldsymbol{\Omega} = -\sum_{i<j} \Omega_{ij}\, e_i \wedge e_j,$$

where

$$e_i \wedge e_j := g(e_i, \cdot)\, e_j - g(e_j, \cdot)\, e_i \tag{4.3}$$

is a basis of $\mathfrak{so}_n$. We then get:

$$\omega_{ij}(X) = -g(\omega(X)e_i, e_j) = -g(\nabla_X e_i, e_j) \tag{4.4}$$

for all $X \in \Gamma(TM)$.

Proposition 4.3 (local description of ∇ and $\mathcal{R}$)**.** 1) *The lifted connection 1–form $\widetilde{\boldsymbol{\omega}}$ satisfies*

$$\widetilde{\boldsymbol{\omega}}(\widetilde{s}_*(X)) := \widetilde{\omega}(X) = -\frac{1}{2} \sum_{i<j} \omega_{ij}(X)\, e_i \cdot e_j. \tag{4.5}$$

2) *Take an orthonormal basis $\sigma_1, \dots, \sigma_N$ of $\Sigma_n \simeq \mathbb{C}^N$ to get a local section $(\psi_\alpha)_{1 \le \alpha \le N}$ in* $\mathrm{SU}(\Sigma M)$ *by*

$$\psi_\alpha := [\widetilde{s}, \sigma_\alpha] \in \Gamma_U(\Sigma M).$$

Then the spinorial covariant derivative is given locally by:

$$\nabla \psi_\alpha = \frac{1}{4} \sum_{i,j=1}^{n} g(\nabla e_i, e_j)\, e_i \cdot e_j \cdot \psi_\alpha\,. \tag{4.6}$$

3) *Finally, if* R *denotes the Riemann curvature tensor of the tangent bundle, then for the Spinorial curvature tensor one gets:*

$$\mathcal{R}_{X,Y}\, \psi = \frac{1}{4} \sum_{i,j=1}^{n} g(\mathrm{R}_{X,Y}\, e_i, e_j)\, e_i \cdot e_j \cdot \psi\,. \tag{4.7}$$

Proof. 1) From (4.2) and Proposition 2.23 follows:

$$\begin{aligned} \widetilde{\omega}(X) &= (\mathrm{Ad}_*^{-1} \circ \boldsymbol{\omega} \circ s_*)(X) \\ &= -\sum_{i<j} \omega_{ij}(X)\, \mathrm{Ad}_*^{-1}(\,(e_i \wedge e_j)\,) \\ &\quad \text{(by (2.23))} \\ &= -\frac{1}{2} \sum_{i<j} \omega_{ij}(X)\, e_i \cdot e_j. \end{aligned}$$

2) This is a direct calculation using $\rho_* = \rho$ (since the representation ρ is a linear map from the Clifford algebra to the vector space of endomorphisms $\mathrm{End}(\Sigma_n)$):

$$\begin{aligned}
\nabla\psi_\alpha &= [\widetilde{s}, \rho_*(\widetilde{\boldsymbol{\omega}} \circ \widetilde{s}_*)\sigma_\alpha], \qquad \text{(see Section 3.2)}\\
&=[\widetilde{s}, \rho_*(-\frac{1}{2}\sum_{i<j}\omega_{ij}\, e_i \cdot e_j)\sigma_\alpha] \qquad \text{(by (4.5))}\\
&= -\frac{1}{2}\sum_{i<j}\omega_{ij}\, e_i \cdot e_j \cdot \psi_\alpha, \qquad \text{(see (2.14))}\\
&=\frac{1}{2}\sum_{i<j} g(\nabla e_i, e_j)\, e_i \cdot e_j \cdot \psi_\alpha \qquad \text{(by (4.4))}\\
&= \frac{1}{4}\sum_{i,j} g(\nabla e_i, e_j)\, e_i \cdot e_j \cdot \psi_\alpha.
\end{aligned}$$

3) This follows directly from Proposition 3.16 and (4.6)

□

Proposition 4.4 (Compatibility of ∇ with "$\cdot$" and $(\cdot,\cdot)$)**.**

$$X(\psi, \varphi) = (\nabla_X\psi, \varphi) + (\psi, \nabla_X\varphi) \tag{4.8}$$

$$\nabla_X(Y \cdot \psi) = (\nabla_X Y) \cdot \psi + Y \cdot \nabla_X\psi \tag{4.9}$$

Proof. Take $s = (e_1, \dots, e_n)$ and ψ_α as in (4.6).

1) For $\psi = \psi_\alpha$ and $\varphi = \psi_\beta$, for any vector field X, one has

$$\begin{aligned}
(\nabla_X\psi_\alpha, \psi_\beta) &\overset{\text{by (4.6)}}{=} \frac{1}{4}\sum_{i,j=1}^{n} g(\nabla_X e_i, e_j)(e_i \cdot e_j \cdot \psi_\alpha, \psi_\beta)\\
&\overset{\text{by (4.1)}}{=} \frac{1}{4}\sum_{i,j=1}^{n} g(\nabla_X e_i, e_j)(\psi_\alpha, e_j \cdot e_i \cdot \psi_\beta)\\
&= \frac{1}{4}\sum_{i,j=1}^{n} g(\nabla_X e_j, e_i)(\psi_\alpha, e_i \cdot e_j \cdot \psi_\beta)\\
&= -\frac{1}{4}\sum_{i,j=1}^{n} g(\nabla_X e_i, e_j)(\psi_\alpha, e_i \cdot e_j \cdot \psi_\beta)
\end{aligned}$$

which combined with the fact that $(\psi_\alpha, \psi_\beta) = \delta_{\alpha\beta}$ yield (4.8). For arbitrary sections, we use bilinearity together with

$$\begin{aligned}
X(f\psi, \varphi) &= (Xf)(\psi, \varphi) + fX(\psi, \varphi)\\
&= (Xf)(\psi, \varphi) + f(\nabla_X\psi, \varphi) + f(\psi, \nabla_X\varphi)\\
&= \big(\nabla_X(f\psi), \varphi\big) + (f\psi, \nabla_X\varphi)
\end{aligned}$$

hence (4.8) is true, if the formula already holds for ψ and φ.

2) First notice that:

$$e_i \cdot e_j \cdot e_k = e_i \cdot (-e_k \cdot e_j - 2\delta_{jk}) = e_k \cdot e_i \cdot e_j + 2\delta_{ik}\, e_j - 2\delta_{jk}\, e_i.$$

Then, for $Y = e_i$ and $\psi = \psi_\alpha$, we get:

$$\begin{aligned}\nabla(e_k \cdot \psi) &= \frac{1}{4}\sum_{i,j} g(\nabla e_i, e_j)\, e_i \cdot e_j \cdot (e_k \cdot \psi_\alpha)\\ &= \frac{1}{4}\sum_{i,j} g(\nabla e_i, e_j)\, e_k \cdot e_i \cdot e_j \cdot \psi_\alpha + \frac{1}{2}\sum_j g(\nabla e_k, e_j)\, e_j \cdot \psi_\alpha\\ &\quad - \frac{1}{2}\sum_i g(\nabla e_i, e_k)\, e_i \cdot \psi_\alpha.\end{aligned}$$

We change i to j in the last term to get

$$\begin{aligned}&= \frac{1}{4}\sum_{i,j} g(\nabla e_i, e_j)\, e_k \cdot e_i \cdot e_j \cdot \psi_\alpha + \sum_j g(\nabla e_k, e_j)\, e_j \cdot \psi_\alpha\\ &= e_k \cdot \nabla\psi_\alpha + (\nabla e_k) \cdot \psi_\alpha.\end{aligned}$$

For arbitrary Y and ψ, formula (4.9) is straightforward as in (4.8). □

4.3. **The Dirac Operator.** In the following we will often use a local orthonormal frame, denoted by $s = (e_1, \ldots, e_n) \in \Gamma_U(\mathrm{SO}\, M)$, $U \subset M$, which yields the relation

$$e_i \cdot e_j + e_j \cdot e_i = -2\delta_{ij} \qquad 1 \le i, j \le n\,. \tag{4.10}$$

In the previous section, we have seen that associated to a spin structure of a Riemannian manifold $(M^n,\, g)$, there are three essential structures:

i) The spinor bundle $\Sigma M = \mathrm{Spin}\, M \times_\rho \Sigma_n$ with the Clifford multiplication

$$\begin{aligned}\mathfrak{m}\colon TM \otimes \Sigma M &\longrightarrow \Sigma M\\ X \otimes \psi &\longmapsto X \cdot \psi := \rho(X)\psi\,,\end{aligned}$$

where ρ is the spinor representation. This multiplication extends to

$$\begin{aligned}\mathfrak{m}\colon \Lambda^p(TM) \otimes \Sigma M &\longrightarrow \Sigma M\\ \alpha \otimes \psi &\longmapsto \sum_{1 \le i_1 < \ldots < i_p \le n} \alpha_{i_1 \ldots i_p}\, e_{i_1} \cdot \ldots \cdot e_{i_p} \cdot \psi,\end{aligned}$$

where locally

$$\alpha = \sum_{1 \le i_1 < \ldots < i_p \le n} \alpha_{i_1 \ldots i_p}\, e^*_{i_1} \wedge \ldots \wedge e^*_{i_p}\,,$$

and $e_i^* = g(e_i, \cdot)$ is the dual basis of e_i.

ii) The natural hermitian product $(\cdot, \cdot)$ on sections of ΣM.

iii) The Levi–Civita connection on ΣM.

As we have seen in (4.1) and Proposition 4.4, these structures satisfy the following compatibility conditions:

$$\begin{aligned} (X \cdot \psi, \varphi) + (\psi, X \cdot \varphi) = 0\,, \\ X(\psi, \varphi) - (\nabla_X \psi, \varphi) - (\psi, \nabla_X \varphi) = 0\,, \\ \nabla_X (Y \cdot \psi) - \nabla_X Y \cdot \psi - Y \cdot \nabla_X \psi = 0\,, \end{aligned}$$

for all X, $Y \in \Gamma(TM)$, ψ, $\varphi \in \Gamma(\Sigma M)$.

We are now able to give the definition of the Dirac operator:

Definition 4.5 (Dirac operator)**.** The Dirac operator is the composition of the covariant derivative acting on sections of ΣM with the Clifford multiplication:

$$\mathcal{D} := \mathfrak{m} \circ \nabla.$$

Locally, we get

$$\begin{array}{cccccc} \mathcal{D} \colon \Gamma(\Sigma M) & \xrightarrow{\nabla} & \Gamma(T^*M \otimes \Sigma M) & \xrightarrow{\mathfrak{m}} & \Gamma(\Sigma M) \\ \psi & \longmapsto & \sum_{i=1}^n e_i^* \otimes \nabla_{e_i} \psi & \longmapsto & \sum_{i=1}^n e_i \cdot \nabla_{e_i} \psi. \end{array}$$

Lemma 4.6. The commutator of the Dirac operator with the action, by pointwise multiplication on the spinor bundle, of a function $f \colon M \to \mathbb{C}$, is given by

$$[\,\mathcal{D}, f\,]\psi := \mathrm{d}f \cdot \psi, \qquad \psi \in \Gamma(\Sigma M).$$

Proof. A local calculation shows

$$\begin{aligned} [\,\mathcal{D}, f\,]\psi = (\mathcal{D}f - f\mathcal{D})\psi &= \sum_{i=1}^n e_i \cdot \nabla_{e_i}(f\psi) - f\mathcal{D}\psi \\ &= \sum_{i=1}^n \mathrm{d}f(e_i)\, e_i \cdot \psi + f\mathcal{D}\psi - f\mathcal{D}\psi = \mathrm{d}f \cdot \psi. \end{aligned}$$

□

Lemma 4.7. The Dirac operator is a first order partial differential operator, which is

i) elliptic and

ii) self-adjoint with respect to $(\cdot, \cdot)_{L^2} := \int_M (\cdot, \cdot)\nu_g$, if M is compact, where ν_g denotes the volume element.

Proof. i) Let $x \in M$, $\xi \in T_x^*M \setminus \{0\}$ and $f \in \mathrm{C}^\infty(M, \mathbb{R})$ such that $(\mathrm{d}f)_x = \xi$, then the *principal symbol*, $\sigma_\xi(\mathcal{D}) : \Sigma_x M \to \Sigma_x M$, is given by

$$\begin{aligned}\sigma_\xi(\mathcal{D})\big(\psi(x)\big) &:= \mathcal{D}\big[(f - f(x))\psi\big](x) \\ &= \big(f\mathcal{D}\psi + \mathrm{d}f \cdot \psi - f(x)\mathcal{D}\psi\big)(x) \\ &= (df)_x \cdot \psi(x) \\ &= \xi \cdot \psi(x),\end{aligned}$$

i.e., $\sigma_\xi(\mathcal{D})$ is Clifford multiplication by ξ. To see that $\mathcal{D}$ is elliptic, we have to check that, for all $\xi \in T^*M\setminus\{0\}$, $\sigma_\xi(\mathcal{D})\colon \Sigma_x M \to \Sigma_x M$ is an isomorphism. But

$$\xi \cdot \psi = 0 \Longrightarrow \xi \cdot \xi \cdot \psi = 0 \Longleftrightarrow -\|\xi\|^2\psi = 0 \Longleftrightarrow \psi = 0.$$

ii) To show that $\mathcal{D}$ is self-adjoint, we choose *normal coordinates* at $x \in M$, i.e. $(\nabla_{e_i} e_j)(x) = 0$, $1 \le i,\ j \le n$, and compute first

$$\begin{aligned}(\mathcal{D}\psi, \varphi) &= (\sum_{i=1}^n e_i \cdot \nabla_{e_i}\psi, \varphi) \\ &= -\sum_{i=1}^n (\nabla_{e_i}\psi, e_i \cdot \varphi) \\ &= -\sum_{i=1}^n \big[e_i(\psi, e_i \cdot \varphi) - (\psi, \nabla_{e_i}(e_i \cdot \varphi))\big] \\ &=\Big|_x -\sum_{i=1}^n e_i(\psi, e_i \cdot \varphi) + (\psi, \mathcal{D}\varphi),\end{aligned}$$

after using (4.8). The sum in the last term then is the divergence of a complex vector field. To see this, consider the two vector fields $X_1,\ X_2 \in \Gamma(TM)$ defined for all $Y \in TM$ by

$$g(X_1, Y) + i\, g(X_2, Y) = (\psi, Y \cdot \varphi)\,.$$

Then, it follows

$$\begin{aligned}\operatorname{div} X_1 + i \operatorname{div} X_2 &= \sum_{k=1}^n g(\nabla_{e_k} X_1, e_k) + i \sum_{l=1}^n g(\nabla_{e_l} X_2, e_l) \\ &= \sum_{k=1}^n \big(e_k\, g(X_1, e_k) - g(X_1, \nabla_{e_k} e_k)\big) \\ &\qquad + i \sum_{l=1}^n \big(e_l\, g(X_2, e_l) - g(X_2, \nabla_{e_l} e_l)\big) \\ &=\Big|_x \sum_{k=1}^n e_k(g(X_1, e_k) + i\, g(X_2, e_k)) \\ &= \sum_{k=1}^n e_k(\psi, e_k \cdot \varphi).\end{aligned}$$

Finally:

$$(\mathcal{D}\psi, \varphi) = -\operatorname{div} X_1 - i \operatorname{div} X_2 + (\psi, \mathcal{D}\varphi).$$

This equation does no longer depend on the choice of the coordinates, so we can integrate over M and get

$$\int_M (\mathcal{D}\psi, \varphi)\nu_g = \int_M (\varphi, \mathcal{D}\psi)\nu_g,$$

since $\partial M = \emptyset$.

□

Lemma 4.8. Let $n = 2m$, then

i) $\mathcal{D}\colon \Gamma(\Sigma^{\pm}M) \to \Gamma(\Sigma^{\mp}M)$, i.e., the Dirac operator sends positive spinors to negative ones and conversely.
ii) The eigenvalues of $\mathcal{D}$ are symmetric with respect to the origin.

Proof. i) Recall that $\Sigma_n^{\pm} := \frac{1}{2}(1 \pm \omega_{\mathbb{C}}) \cdot \Sigma_n$ (Proposition 2.25), so Σ_n^{+} is the subspace on which multiplication with $\omega_{\mathbb{C}}$ is the identity, and Σ_n^{-} the one on which the multiplication with $\omega_{\mathbb{C}}$ is minus the identity. We therefore get for $\psi^+ \in \Gamma(\Sigma^+ M)$:

$$\begin{aligned}\omega_{\mathbb{C}} \cdot \mathcal{D}\psi^+ = \omega_{\mathbb{C}} \cdot \sum_{i=1}^{n} e_i \cdot \nabla_{e_i}\psi^+ &= -\sum_i e_i \cdot \omega_{\mathbb{C}} \cdot \nabla_{e_i}\psi^+ \\ &= -\sum_{i=1}^{n} e_i \cdot \nabla_{e_i}(\omega_{\mathbb{C}} \cdot \psi^+) = -\mathcal{D}\psi^+.\end{aligned}$$

ii) Let ψ be an eigenspinor for $\mathcal{D}$, i.e. $\mathcal{D}\psi = \lambda\psi$ for $\lambda \in \mathbb{R}$ and decompose $\psi = \psi^+ + \psi^-$. Then $\mathcal{D}\psi^+ + \mathcal{D}\psi^- = \lambda\psi^- + \lambda\psi^+$, yields with i): $\mathcal{D}\psi^{\pm} = \lambda\psi^{\mp}$. So the spinor $\widetilde{\psi} := \psi^+ - \psi^-$ is an eigenspinor of $\mathcal{D}$ associated with the eigenvalue $-\lambda$, since

$$\mathcal{D}\widetilde{\psi} = \mathcal{D}(\psi^+ - \psi^-) = \lambda\psi^- - \lambda\psi^+ = -\lambda(\psi^+ - \psi^-) = -\lambda\widetilde{\psi}.$$

□

Examples 4.9. i) Let $M = \mathbb{R}^n$, $\Sigma\mathbb{R}^n = \mathbb{R}^n \times \mathbb{C}^N$, $N = 2^{[\frac{n}{2}]}$, then every spinorfield $\psi \in \Gamma(\Sigma\mathbb{R}^n)$ is in fact a map $\psi\colon \mathbb{R}^n \to \mathbb{C}^N$, and the Dirac operator is given by

$$\mathcal{D} = \sum_{i=1}^{n} e_i \cdot \partial_i,$$

which acts on differentiable maps from $\mathbb{R}^n$ to $\mathbb{C}^N$, where $\partial_i = \nabla_{e_i}$. Then

$$\begin{aligned}
\mathcal{D}^2 &= \left(\sum_{i=1}^{n} e_i \cdot \partial_i\right)\left(\sum_{j=1}^{n} e_j \cdot \partial_j\right) \\
&= \sum_{i,j} e_i \cdot e_j \cdot \partial_i \partial_j \\
&= -\sum_{i} \partial_i^2 + \sum_{i<j} e_i \cdot e_j \cdot \partial_i \partial_j + \sum_{i>j} e_i \cdot e_j \cdot \partial_i \partial_j \\
&= -\sum_{i} \partial_i^2 + \sum_{i<j} e_i \cdot e_j \cdot \partial_i \partial_j + \sum_{i<j} e_j \cdot e_i \cdot \partial_j \partial_i \\
&= -\sum_{i} \partial_i^2 + \sum_{i<j} e_i \cdot e_j \cdot \Big(\partial_i \partial_j - \partial_j \partial_i\Big) \\
&= -\sum_{i=1}^{n} \partial_i^2 = \begin{pmatrix} \Delta & & 0 \\ & \ddots & \\ 0 & & \Delta \end{pmatrix}.
\end{aligned}$$

ii) In the two dimensional case $M = \mathbb{R}^2$, we have $\mathbb{C}l_2 = \mathbb{C}(2)$ and the decomposition $\Sigma_2 = \Sigma_2^+ \oplus \Sigma_2^- = \mathbb{C} \oplus \mathbb{C}$, which is $\Sigma_2^+ = \operatorname{span}_{\mathbb{C}}(e_1 + ie_2)$ and $\Sigma_2^- = \operatorname{span}_{\mathbb{C}}(1 - ie_1 \cdot e_2)$. Then, each spinor field $\psi \in \Gamma(\Sigma M)$ is given by two complex functions f, $g\colon \mathbb{R}^2 \to \mathbb{C}$, such that

$$\psi = f(e_1 + ie_2) + g(1 - ie_1 \cdot e_2).$$

The Dirac operator now becomes

$$\begin{aligned}
\mathcal{D}\psi &= (e_1 \cdot \partial_1 + e_2 \cdot \partial_2)[(e_1 + ie_2)f + (1 - ie_1 \cdot e_2)g] \\
&= -(\partial_1 + i\partial_2)f\,(1 - ie_1 \cdot e_2) + (\partial_1 - i\partial_2)g\,(e_1 + ie_2) \\
&= 2(-\partial_{\overline{z}} f\,(1 - e_1 \cdot e_2) + \partial_z g\,(e_1 + ie_2)),
\end{aligned}$$

where $\partial_{\overline{z}} = \frac{1}{2}(\partial_1 + i\partial_2)$ and $\partial_z = \frac{1}{2}(\partial_1 - i\partial_2)$. That is

$$\mathcal{D} = \begin{pmatrix} 0 & 2\partial_z \\ -2\partial_{\overline{z}} & 0 \end{pmatrix}$$

in the basis $\{(e_1 + ie_2), (1 - ie_1 \cdot e_2\}$ of Σ_2. Hence the Dirac operator $\mathcal{D}$ could be considered as a generalization of the Cauchy-Riemann operator.

iii) The *Clifford bundle* ClM.

For a Riemannian manifold (M^n, g), define the vector bundle ClM by

$$(ClM)_x = \text{the Clifford algebra of } (T_xM, g_x).$$

We can see this bundle as an associated vector bundle to $\mathrm{SO}\,M$. By the universal property (2.5), one can extend

$$\rho_n\colon \mathrm{SO}_n \to \mathrm{SO}(\mathbb{R}^n) \qquad \text{to} \qquad \widetilde{\rho}_n\colon \mathrm{SO}_n \to \mathrm{Aut}(Cl_n),$$

so that

$$ClM = \mathrm{SO}\,M \times_{\widetilde{\rho}_n} Cl_n.$$

From (2.8) we have

$$v \cdot \varphi \simeq v \wedge \varphi - v \lrcorner \varphi$$

under the isomorphism $Cl(T_xM, g_x) \xrightarrow{\simeq} \Lambda^*(T_xM)$. The differential and its adjoint could be locally written as

$$\mathrm{d} = \sum_{i=1}^{n} e_i \wedge \nabla_{e_i} \qquad \text{and} \qquad \delta = -\sum_{i=1}^{n} e_i \lrcorner \nabla_{e_i}.$$

If we define the Dirac operator as before, we then have

$$\mathcal{D} := \sum_{i=1}^{n} e_i \cdot \nabla_{e_i} \simeq \mathrm{d} + \delta.$$

This is the "square root" of the Laplacian, since on $\Lambda^*(TM)$

$$\mathcal{D}^2 \simeq (\mathrm{d}+\delta)^2 = \mathrm{d}\delta + \delta\mathrm{d} = \Delta\,.$$

4.4. **The Schrödinger-Lichnerowicz Formula.**

Definition 4.10 (Extension of $(\cdot,\cdot)$ and ∇)**.**

i) Extend the natural Hermitian product $(\cdot,\cdot)$ on $\Gamma(\Sigma M)$ to sections of $T^*M \otimes \Sigma M$ by

$$\begin{array}{rcl} (\cdot,\cdot)\colon \Gamma(T^*M \otimes \Sigma M) \times \Gamma(T^*M \otimes \Sigma M) & \longrightarrow & C^\infty(M, \mathbb{C}) \\ \big(\alpha \otimes \psi, \beta \otimes \varphi\big) & \longmapsto & g(\alpha,\beta)\,(\psi,\varphi), \end{array}$$

where the metric g extends to covectors by the isomorphism $T^*M \simeq TM$ induced by g. Thus, for ω, $\eta \in \Gamma(T^*M \otimes \Sigma M)$, we get

$$(\omega,\eta) = \sum_{i=1}^{n} (\omega(e_i), \eta(e_i))$$

for any orthonormal basis $\{e_1,\dots,e_n\}$ in T_xM.

ii) Let ∇^* be the formal adjoint of ∇, i.e.

$$\nabla^*\colon \Gamma(T^*M \otimes \Sigma M) \longrightarrow \Gamma(\Sigma M)$$

such that $(\nabla^*\Psi, \varphi)_{L^2} = (\Psi, \nabla\varphi)_{L^2}$ for every $\Psi \in \Gamma(T^*M \otimes \Sigma M)$ and $\varphi \in \Gamma(\Sigma M)$.

Lemma 4.11. In local normal coordinates $(e_1, \dots, e_n)$ at $x \in M$, we have:

$$\nabla^* \nabla \psi = -\sum_{i=1}^{n} \nabla_{e_i} \nabla_{e_i} \psi$$

for all $\psi \in \Gamma(\Sigma M)$.

Proof.

$$(\nabla^* \nabla \psi, \varphi)_{L_2} = (\nabla \psi, \nabla \varphi)_{L^2} = \sum_{i=1}^{n} (\nabla_{e_i} \psi, \nabla_{e_i} \varphi)_{L^2}$$

As in the proof of Lemma 4.7 ii), we get:

$$\begin{aligned}
\sum_{i=1}^{n} (\nabla_{e_i} \psi, \nabla_{e_i} \varphi) &= \sum_{i=1}^{n} e_i (\nabla_{e_i} \psi, \varphi) - (\nabla_{e_i} \nabla_{e_i} \psi, \varphi) \\
&= \operatorname{div} X_1 + \operatorname{div} X_2 - \sum_{i=1}^{n} (\nabla_{e_i} \nabla_{e_i} \psi, \varphi),
\end{aligned}$$

which, by integration, gives the required condition for ∇^* to be the formal adjoint of ∇. □

Proposition 4.12. *Let $\widetilde{\mathcal{R}} := \frac{1}{2} \sum_{i,j=1}^{n} e_i \cdot e_j \cdot \mathcal{R}_{e_i, e_j}$, where $\mathcal{R}$ is the spinorial curvature. Then we get for the square of the Dirac operator:*

$$\mathcal{D}^2 = \nabla^* \nabla + \widetilde{\mathcal{R}}.$$

Proof. Take normal coordinates at $x \in M$, then

$$\begin{aligned}
\mathcal{D}^2 &= \left(\sum_{i=1}^{n} e_i \cdot \nabla_{e_i} \right) \left(\sum_{j=1}^{n} e_j \cdot \nabla_{e_j} \right) =\Big|_x \sum_{i,j=1}^{n} e_i \cdot e_j \cdot \nabla_{e_i} \nabla_{e_j} \\
&= -\sum_{i=1}^{n} \nabla_{e_i} \nabla_{e_i} + \sum_{\substack{i,j=1 \\ i \neq j}}^{n} e_i \cdot e_j \cdot \nabla_{e_i} \nabla_{e_j} \\
&= -\sum_{i=1}^{n} \nabla_{e_i} \nabla_{e_i} + \sum_{i<j}^{n} e_i \cdot e_j \cdot \left(\nabla_{e_i} \nabla_{e_j} - \nabla_{e_j} \nabla_{e_i} \right) \\
&= \nabla^* \nabla + \frac{1}{2} \sum_{i,j=1}^{n} e_i \cdot e_j \ \cdot \mathcal{R}_{e_i, e_j} \\
&= \nabla^* \nabla + \widetilde{\mathcal{R}}.
\end{aligned}$$

□

Theorem 4.13 (The Schrödinger-Lichnerowicz formula)**.** *Let S be the scalar curvature of M, then*

$$\mathcal{D}^2 = \nabla^* \nabla + \frac{1}{4} S \operatorname{Id}_{\Gamma(\Sigma M)}.$$

Proof. By Proposition 4.12, it is sufficient to show that $\widetilde{\mathcal{R}} = \frac{1}{4}S\,\mathrm{Id}_{\Gamma(\Sigma_M)}$. Let Ric be the Ricci tensor of the Riemann tensor R and use Proposition 4.3 for $\mathcal{R}$, to get

$$\begin{aligned}
\widetilde{\mathcal{R}} &= \frac{1}{2}\sum_{i,j} e_i \cdot e_j \cdot \mathcal{R}_{e_i,e_j} = \frac{1}{2}\sum_{i,j} e_i \cdot e_j \cdot \left(\frac{1}{4}\sum_{k,l} g(\mathrm{R}_{e_i,e_j}\, e_k, e_l) e_k \cdot e_l\right) \\
&= \frac{1}{8}\sum_{l}\left(\sum_{i,j,k} \mathrm{R}_{ijkl}\, e_i \cdot e_j \cdot e_k\right) \cdot e_l \\
&= \frac{1}{8}\sum_{l}\left(\frac{1}{3}\sum_{i\neq j\neq k} (\mathrm{R}_{ijkl} + \mathrm{R}_{jkil} + \mathrm{R}_{kijl})\ e_i \cdot e_j \cdot e_k \right.\\
&\qquad \left. + \sum_{i,j} g(\mathrm{R}_{e_i,e_j}\, e_i, e_l)\ e_i \cdot e_j \cdot e_i + \sum_{i,j} g(\mathrm{R}_{e_i,e_j}\, e_j, e_l)\ e_i \cdot e_j \cdot e_j\right) \cdot e_l \\
&= \frac{1}{8}\sum_{i,j,l}\Big[g(\mathrm{R}_{e_i,e_j}\, e_i, e_l)\ e_j - g(\mathrm{R}_{e_i,e_j}\, e_j, e_l) e_i\Big] \cdot e_l \\
&= \frac{1}{8}\left(\sum_{j,l} - \mathrm{Ric}(e_j, e_l)\ e_j \cdot e_l - \sum_{i,l} \mathrm{Ric}(e_i, e_l)\ e_i \cdot e_l\right) \\
&= -\frac{1}{4}\sum_{i,j} \mathrm{Ric}(e_i, e_j)\ e_i \cdot e_j \\
&= \frac{1}{4}\, S
\end{aligned}$$

where we have used $e_i \cdot e_j \cdot e_k = e_j \cdot e_k \cdot e_i = e_k \cdot e_i \cdot e_j$ for $i \neq j \neq k$. $\square$

5. Spectral Properties of the Dirac Operator

We are now ready to study the spectral properties of the Dirac operator. For this, we start by showing that Friedrich inequality could be obtained as a corollary of the Cauchy-Schwarz inequality. We then examine the limiting case which is charaterized by the existence of a Killing spinor. In this set up, the Twistor equation and the Twistor operator appear naturally. The conformal covariance of the Dirac operator is then used to relate its first eigenvalue to that of the scalar conformal Laplacian.

The last subsection is devoted to give a brief presentation, without proofs, of recent results concerning the gap in the spectrum of the Dirac operator for different holonomies. For a complete overview with proofs, we refer to [BHMM].

5.1. The First Eigenvalue of the Dirac Operator.

Theorem 5.1. [Li1] *On an n-dimensional compact Riemannian spin manifold (M^n, g) with positive scalar curvature S we have*

i) $\ker \mathcal{D} = \{0\}$.
ii) *If* $\mathcal{D}\psi = \lambda\psi$ *for a non–trivial spinor* $\psi \in \Gamma(\Sigma M)$*, then* $\lambda^2 > \frac{1}{4}S_0$*, where* $S_0 := \min_M S$*.*

Proof. i) By the Schrödinger-Lichnerowicz formula 4.13, for any non-trivial
spinor field $\psi \in \Gamma(\Sigma M)$, we have

$$\int_M |\mathcal{D}\psi|^2 \nu_g = \int_M \underbrace{|\nabla\psi|^2}_{\geq 0} \nu_g + \int_M \underbrace{\frac{1}{4}S|\psi|^2}_{>0} \nu_g .$$

Thus $\mathcal{D}\psi$ cannot be identically zero.
ii) Let $\mathcal{D}\psi = \lambda\psi$ for a non–trivial spinor $\psi \in \Gamma(\Sigma M)$. Then

$$\int_M |\mathcal{D}\psi|^2 \nu_g - \frac{1}{4}\int_M S|\psi|^2 \nu_g = \int_M |\nabla\psi|^2 \nu_g \geq 0.$$

Hence, $\lambda^2 - \frac{1}{4}S_0 \geq 0$. If $\lambda^2 - \frac{1}{4}S_0 = 0$, then $\nabla\psi = 0$ and therefore $\mathcal{D}\psi = 0$, which by i) is impossible.

□

Remark 5.2. On a compact Riemannian spin manifold, if the scalar curvature is positive, then

$$\operatorname{Ind}_A \mathcal{D}^+ := \dim\ker \mathcal{D}^+ - \dim\ker \mathcal{D}^- = 0\,.$$

By the Atiyah–Singer Index theorem, this analytical index is equal to the topological index, which yields a topological obstruction for the existence of positive scalar curvature metrics. In this setup, another beautiful application of the Index theorem is given in [VW] where qualitative informations on the gap between the eigenvalues of the Dirac operator are obtained.

Theorem 5.3. [Fr1] *Given a compact Riemannian spin manifold* (M^n, g)*, then for any eigenvalues* λ *of* $\mathcal{D}$*,*

$$\lambda^2 \geq \frac{1}{4}\frac{n}{n-1}S_0\,,$$

for all eigenvalues λ *of* $\mathcal{D}$*, where* $S_0 := \min_M S$ *as before.*

Proof. The proof we give here is due to S. Gallot [Ga2]. It is based on the Cauchy-Schwarz Inequality. For an arbitrary spinor $\psi \in \Gamma(\Sigma M)$ we have

$$\begin{aligned} |\mathcal{D}\psi|^2 &= \Big|\sum_{i=1}^n e_i \cdot \nabla_{e_i}\psi\Big|^2 \leq \Big(\sum_{i=1}^n |e_i \cdot \nabla_{e_i}\psi|\Big)^2 \\ &= \Big(\sum_{i=1}^n 1\,|\nabla_{e_i}\psi|\Big)^2 \leq n\sum_{i=1}^n |\nabla_{e_i}\psi|^2 = n\,|\nabla\psi|^2. \end{aligned} \tag{5.1}$$

By the Schrödinger-Lichnerowicz formula we get

$$\int_M \frac{1}{n}|\mathcal{D}\psi|^2\,\nu_g \leq \int_M |\nabla\psi|^2\,\nu_g = \int_M |\mathcal{D}\psi|^2\,\nu_g - \frac{1}{4}\int_M S|\psi|^2\,\nu_g$$

$$\Longrightarrow\ (1-\frac{1}{n})\int_M |\mathcal{D}\psi|^2\,\nu_g \geq \frac{1}{4}\int_M S|\psi|^2\,\nu_g.$$

For a non-trivial eigenspinor $\psi \in \Gamma(\Sigma M)$, it follows

$$\lambda^2\int_M |\psi|^2\,\nu_g \geq \frac{1}{4}\frac{n}{n-1}\int_M S|\psi|^2\,\nu_g \geq \frac{1}{4}\frac{n}{n-1}S_0\int_M |\psi|^2\,\nu_g.$$

□

Remark 5.4. If $\psi \in \Gamma(\Sigma M)$ is an eigenspinor for which equality holds, that is

$$\lambda^2 = \frac{1}{4}\frac{n}{n-1}S_0$$

Then ψ satisfies the *twistor equation*

$$\nabla_X\psi + \frac{1}{n}X\cdot\mathcal{D}\psi = 0,$$

for all $X \in TM$, which specializes to the *Killing equation*

$$\nabla_X\psi + \frac{\lambda}{n}X\cdot\psi = 0\,, \qquad \forall X \in \Gamma(TM)\,, \tag{5.2}$$

since ψ is an eigenspinor.

Proof. In case of equality in (5.1), we get by Cauchy–Schwarz

$$e_i\cdot\nabla_{e_i}\psi = e_j\cdot\nabla_{e_j}\psi, \qquad \forall i,j, 1\leq i,j\leq n\,,$$

which is equivalent to

$$\forall i,\ 1\leq i\leq n, \qquad \mathcal{D}\psi = ne_i\cdot\nabla_{e_i}\psi \Longleftrightarrow \nabla_{e_i}\psi + \frac{1}{n}e_i\cdot\mathcal{D}\psi = 0\,.$$

□

Definition 5.5. For a Riemannian spin manifold the *twistor operator* is defined by

$$\mathcal{P}\colon \Gamma(\Sigma M) \xrightarrow{\nabla} \Gamma(T^*M\otimes\Sigma M) \xrightarrow{\pi} \Gamma(\ker\mathfrak{m}) \tag{5.3}$$

where π is the orthogonal projection on the kernel of the Clifford multiplication $\mathfrak{m}$.

Recall that the Dirac operator is defined by $\mathcal{D} := \mathfrak{m}\circ\nabla$. In fact, the twistor operator is the complementary of the Dirac operator in the following sense:

$$\begin{tikzcd} \Gamma(\Sigma M) \arrow[r, "\nabla"] & \Gamma(T^*M\otimes\Sigma M) \arrow[r, "\mathfrak{m}"] \arrow[dr, "\pi"'] & \Gamma(\Sigma M) \\ & & \Gamma(\ker\mathfrak{m}) \end{tikzcd} \tag{5.4}$$

where the composition above is the Dirac operator $\mathcal{D}$ and the composition below, is the twistor operator $\mathcal{P}$.

Remark 5.6. In fact (5.4) is given by the composition of the covariant derivative with the projections on the Spin_n-irreducible components of $\Gamma(T^*M \otimes \Sigma M)$. This is a special case of a general situation developed in [SW]. In [Fe], it is shown that all such differential operators are conformally covariant (see next section where this fact is proved for the Dirac operator). We refer to [Br1, Br2] for further results in this direction, based on a systematic use of representation theory.

Proposition 5.7. *In local coordinates we have*

$$\mathcal{P}\,\psi = \sum_i e_i \otimes \big(\nabla_{e_i}\psi + \frac{1}{n} e_i \cdot \mathcal{D}\psi\big).$$

Proof. The projection π is given by

$$v \otimes \psi \mapsto v \otimes \psi + \frac{1}{n}\sum_i e_i \otimes e_i \cdot \mathfrak{m}(v \otimes \psi)\,,$$

since one easily sees, that $\ker \mathfrak{m} = \operatorname{Im}\pi$, $\pi^2 = \pi$ and π is selfadjoint. □

Remark 5.8. Furthermore, for $\alpha \in (\ker \mathfrak{m})^\perp = \ker \pi$ we have

$$(\mathfrak{m}(\alpha), \mathfrak{m}(\alpha)) = n(\alpha, \alpha)$$

since $\alpha = -\frac{1}{n}\sum_i e_i \otimes e_i \cdot \mathfrak{m}(\alpha)$, which yields by Pythagoras theorem to

$$|\nabla\psi|^2 = \frac{1}{n}|\mathcal{D}\psi|^2 + |\,\mathcal{P}\,\psi|^2.$$

Definition 5.9. A non trivial spinor $\psi \in \Gamma(\Sigma M)$ is called

i) a *twistor spinor* if $\mathcal{P}\,\psi = 0$, i.e.

$$\forall X \in \Gamma(TM), \qquad \nabla_X\psi + \frac{1}{n}X \cdot \mathcal{D}\psi = 0\,. \tag{5.5}$$

ii) a *generalized Killing spinor* if $\mathcal{P}\,\psi = 0$ and $\mathcal{D}\psi = f\psi$ for some $f \in \mathrm{C}^\infty(M)$, i.e.

$$\nabla_X\psi + \frac{f}{n}X \cdot \psi = 0\,. \tag{5.6}$$

for all $X \in \Gamma(TM)$.

If $\psi \in \Gamma(\Sigma M)$ is a Killing spinor, then its *associated vector field* is defined by

$$X_\psi := i\sum_{j=1}^{n}(\psi, e_j \cdot \psi)e_j.$$

(The function $i(\psi, e_j\psi)$ is real, because generally for a k–form α we have $(\alpha \cdot \psi, \psi) = (-1)^{\frac{k(k+1)}{2}}\,\overline{(\alpha \cdot \psi, \psi)}$).

Lemma 5.10. For a Killing spinor ψ, the associated vector field X_ψ is a *Killing vector field*, i.e. $\pounds_X\, g = 0$ – the Lie derivative of the metric in the direction of X vanishes, hence the name for such spinors.

Proof. We need the following formula for arbitrary vector fields X, Y, Z

$$\begin{aligned}(\pounds_X\, g)(Y,Z) &:= Xg(Y,Z) - g(\pounds_X\, Y,Z) - g(Y,\pounds_X\, Z)\\ &= g(\nabla_X Y,Z) + g(Y,\nabla_X Z) - g([X,Y],Z) - g(Y,[X,Z])\\ &= g(\nabla_Y X,Z) + g(Y,\nabla_Z X)\end{aligned}$$

since ∇ is metric and torsion free. On the other hand, by definition of the associated vector field, in normal coordinates at the point $x \in M$, it follows

$$\begin{aligned}\nabla_Y X_\psi &= i\sum_{j=1}^n \Big[(\nabla_Y\psi, e_j\cdot\psi)e_j + (\psi, e_j\cdot\nabla_Y\psi)\Big]e_j\\ &= -i\,\frac{f}{n}\sum_{j=1}^n(\psi,(e_j\cdot Y - Y\cdot e_j)\cdot\psi)e_j\end{aligned}$$

since ψ is a Killing spinor. Now, for any vector fields Y and Z, we get

$$\begin{aligned}\big(\pounds_{X_\psi}\, g\big)(Y,Z) &= -i\frac{f}{n}\Big((\psi,(Z\cdot Y - Y\cdot Z)\cdot\psi)\\ &\qquad + (\psi,(Y\cdot Z - Z\cdot Y)\cdot\psi)\Big)\\ &= 0\end{aligned}$$

□

Another important property of Killing spinors is the following:

Lemma 5.11. The function $|\psi|^2$ corresponding to a Killing spinor ψ is constant. Hence, a non-trivial Killing spinor has no zeros.

Proof. For this, we prove that for any vector field X, the Lie derivative of $|\psi|^2$ in the direction of X is zero. By (4.8), (5.6) and (4.1), it follows

$$\begin{aligned}X|\psi|^2 &= (\nabla_X\psi,\psi) + (\psi,\nabla_X\psi)\\ &= -\frac{f}{n}\Big((X\cdot\psi,\psi) + (\psi, X\cdot\psi)\Big) = 0\,.\end{aligned}$$

□

We now examine some necessary conditions for the existence of Killing spinors.

Proposition 5.12. *Let $\psi \in \Gamma(\Sigma M)$ be a generalized Killing spinor (see (5.6). Then,*

i) *f is constant and $f^2 = \frac{n}{4(n-1)}S$, and*
ii) *(M^n, g) is an Einstein manifold, i.e.* $\mathrm{Ric} = \frac{S}{n}\,g$.

Proof. Since ψ is a Killing spinor, we have

$$\begin{aligned}
\nabla_Y\psi &= -\frac{f}{n}Y\cdot\psi \\
\nabla_X\nabla_Y\psi &= -\frac{1}{n}X(f)Y\cdot\psi - \frac{f}{n}Y\cdot\nabla_X\psi - \frac{f}{n}(\nabla_X Y)\cdot\psi \\
&= -\frac{1}{n}X(f)Y\cdot\psi + \frac{f^2}{n^2}Y\cdot X\cdot\psi - \frac{f}{n}(\nabla_X Y)\cdot\psi\,; \\
\nabla_Y\nabla_X\psi &= -\frac{1}{n}Y(f)X\cdot\psi + \frac{f^2}{n^2}X\cdot Y\cdot\psi - \frac{f}{n}(\nabla_Y X)\cdot\psi \\
\mathcal{R}_{X,Y}\,\psi &= -\frac{1}{n}\big(X(f)Y - Y(f)X\big)\cdot\psi + \frac{f^2}{n^2}\big(Y\cdot X - X\cdot Y\big)\cdot\psi.
\end{aligned}$$

Using the first Bianchi identity and Equation 3) of Proposition 4.3, we get

$$-\frac{1}{2}\operatorname{Ric}(X)\cdot\psi = \sum_{i=1}^{n} e_i\cdot\mathcal{R}_{X,e_i}\,\psi.$$

Then we compute

$$\begin{aligned}
-\frac{1}{2}\operatorname{Ric}(X)\cdot\psi &= \sum_i e_i\cdot\mathcal{R}_{X,e_i}\,\psi \\
&= -\frac{1}{n}\sum_i\big(X(f)e_i\cdot e_i\cdot\psi - e_i(f)e_i\cdot X\cdot\psi\big) \\
&\quad + \frac{f^2}{n^2}\sum_i(e_i\cdot e_i\cdot X - e_i\cdot X\cdot e_i)\cdot\psi\,.
\end{aligned}$$

Using the fact that $e_i\cdot e_i = -1$ and $\sum_i e_i\cdot X\cdot e_i = (n-2)X$, it follows

$$\begin{aligned}
-\frac{1}{2}\operatorname{Ric}(X)\cdot\psi &= -\frac{1}{n}\big(-nX(f)\cdot\psi - \mathrm{d}f\cdot X\cdot\psi\big) \\
&\quad + \frac{f^2}{n^2}\Big(-nX - (n-2)X\Big)\cdot\psi \\
&= X(f)\psi + \frac{1}{n}\,\mathrm{d}f\cdot X\cdot\psi - \frac{2(n-1)}{n^2}f^2\,X\cdot\psi. \qquad (5.7)
\end{aligned}$$

Since for all vector X and any spinor ψ, $(X\cdot\psi,\psi) = -\overline{(X\cdot\psi,\psi)}$, this function is purely imaginary. For $X := \operatorname{grad} f$, it then follows

$$\begin{aligned}
(-\frac{1}{2}\operatorname{Ric}(X)\cdot\psi + \frac{2(n-1)}{n^2}f^2X\cdot\psi,\psi) &= X(f)|\psi|^2 + \frac{1}{n}(\mathrm{d}f\cdot X\cdot\psi,\psi) \\
&= \mathrm{d}f(X)|\psi|^2 - \frac{1}{n}|\mathrm{d}f|^2|\psi|^2 \\
&= (1-\frac{1}{n})|\mathrm{d}f|^2|\psi|^2\,,
\end{aligned}$$

the l.h.s. being purely imaginary, thus $|\mathrm{d}f|^2 = 0$, since by Lemma 5.11, $|\psi|^2 \neq 0$. This yields

$$X(f)\psi = \frac{1}{n}\mathrm{d}f \cdot X \cdot \psi = 0, \qquad \forall X \in \Gamma(TM),$$

which when combined with (5.7), yield

$$\mathrm{Ric}(X) = \frac{4(n-1)}{n^2} f^2 X$$

Thus $\frac{4(n-1)}{n^2} f^2 = \frac{S}{n}$, by definition of the scalar curvature. $\square$

5.2. **Conformal Covariance of the Dirac Operator.** Let (M,g) be a Riemannian spin manifold. For a conformal change of the metric

$$\overline{g} := e^{2u} g, \qquad u \colon M \to \mathbb{R},$$

the spin structure on (M^n, g) induces a spin structure on $(M^n, \overline{g})$. An isomorphism of the two SO_n–principal fibre bundles is given by

$$\begin{array}{rccc} G_u \colon & \mathrm{SO}_g M & \longrightarrow & \mathrm{SO}_{\overline{g}} M \\ & (e_1, \dots, e_n) & \longmapsto & (e^{-u} e_1, \dots, e^{-u} e_n). \end{array}$$

The spin structure induced on $(M, \overline{g})$ is defined up to isomorphism by the commutative diagram

$$\begin{tikzcd} \mathrm{Spin}_g M \arrow[r, "\widetilde{G}_u"] \arrow[d, "\eta_g"'] & \mathrm{Spin}_{\overline{g}} M \arrow[d, "\eta_{\overline{g}}"] \\ \mathrm{SO}_g M \arrow[r, "G_u"] & \mathrm{SO}_{\overline{g}} M \end{tikzcd}$$

All the arrows in the diagram are supposed to be invariant under the group action. Let $\rho\colon \mathrm{Spin}_n \to \mathrm{Aut}(\Sigma)$ be the spinor representation. An isomorphism of the associated spinor bundles is explicitly given by

$$\begin{array}{rcl} \widetilde{\mathbf{G}}_u \colon \Sigma M = \mathrm{Spin}_g M \times_\rho \Sigma & \longrightarrow & \Sigma\overline{M} = \mathrm{Spin}_{\overline{g}} M \times_\rho \Sigma \\ \psi = [s, \sigma] & \longmapsto & \overline{\psi} = [\widetilde{G}_u(s), \sigma]. \end{array}$$

This map is an isometry with respect to the Hermitian product on the spinor bundles. Together with the corresponding isometry of the tangent bundle, given by $X \mapsto \overline{X} := e^{-u} X$, we have the following relation between " $\cdot$ " and " $\bar{\cdot}$ "

$$\overline{X} \,\bar{\cdot}\, \overline{\psi} = \overline{X \cdot \psi}$$

for every $X \in \Gamma(TM)$ and $\psi \in \Gamma(\Sigma M)$.

Proposition 5.13. [Hn, Ba] *Let (M,g) be a Riemannian spin manifold and $\overline{g} = e^{2u} g$ a conformal change of the metric, then one has*

$$\overline{\mathcal{D}}(e^{-\frac{n-1}{2}u}\, \overline{\psi}) = e^{-\frac{n+1}{2}u}\, \overline{\mathcal{D}\psi}$$

for every $\psi \in \Gamma(\Sigma M)$.

Remark 5.14. It follows therefore, that the dimension of the space of harmonic spinors $H = \{\psi \in \Gamma(\Sigma M) \mid \mathcal{D}\psi = 0\}$ is conformally invariant. In contrast to the Laplace–Beltrami operator on p–forms, the dimension of H for $\mathcal{D}$ is not topologically invariant (see [Hn]).

Proposition 5.15. *Let (M, g) be a Riemannian spin manifold and a conformal change of the metric given by $\overline{g} = e^{2u}g$. Then*

$$\overline{\nabla}_X\overline{\psi} = \overline{\nabla_X\psi} - \frac{1}{2}\,\overline{X\cdot \mathrm{d}u\cdot\psi} - \frac{1}{2}\,X(u)\overline{\psi}$$

for all $\psi \in \Gamma(\Sigma M)$, $X \in \Gamma(TM)$.

Proof. By the Koszul formula and the fact that ∇ and $\overline{\nabla}$ are torsion free, we have in a local orthonormal basis $(e_1, \dots, e_n)$

$$\overline{\omega}_{ij}(e_k) = \omega_{ij}(e_k) + e_i(u)\delta_{jk} - e_j(u)\delta_{ik}, \qquad \forall i, j, k = 1, \dots n \tag{5.8}$$

Let $(\psi_1, \dots, \psi_N)$ be a local section of $\mathrm{Spin}_g M$, such that $\psi_\alpha = [s, \sigma_\alpha]$ for a basis $(\sigma_1, \dots, \sigma_N)$ of Σ. Then by (4.5)

$$\begin{aligned}\overline{\nabla}_{e_k}\overline{\psi}_\alpha &= \frac{1}{4}\sum_{i,j}\overline{\omega}_{ij}(e_k)\,\overline{e_i\cdot e_j\cdot\psi_\alpha}\\ &\overset{(5.8)}{=} \overline{\nabla_{e_k}\psi_\alpha} - \frac{1}{2}\,\overline{e_k\cdot \mathrm{grad}(u)\cdot\psi_\alpha} - \frac{1}{2}\,e_k(u)\overline{\psi}_\alpha\end{aligned}$$

for all $\alpha = 1, \dots, N$. □

Lemma 5.16. The Dirac operator associated with the metric $\overline{g}$ satisfies

$$\overline{\mathcal{D}}(f\,\overline{\psi}) = f\,\overline{\mathcal{D}}\,\overline{\psi} + e^{-u}\,\overline{\mathrm{d}f\cdot\psi}\,.$$

Proof.

$$\begin{aligned}\overline{\mathcal{D}}(f\,\overline{\psi}) &= \sum_i \overline{e}_i\,\bar{\cdot}\,\overline{\nabla}_{\overline{e}_i}(f\,\overline{\psi}) = \sum_i \overline{e}_i\,\bar{\cdot}\left[\overline{e}_i(f)\overline{\psi} + f\,\overline{\nabla}_{\overline{e}_i}\overline{\psi}\right]\\ &= f\,\overline{\mathcal{D}}\,\overline{\psi} + e^{-u}\,\overline{\mathrm{d}f\cdot\psi}.\end{aligned}$$

Recall that $\overline{e}_i = e^{-u}\,e_i$. □

Lemma 5.17. The Dirac operators $\mathcal{D}$ and $\overline{\mathcal{D}}$ are related by

$$\overline{\mathcal{D}}\,\overline{\psi} = e^{-u}\left[\overline{\mathcal{D}\psi} + \frac{n-1}{2}\,\overline{\mathrm{d}u\cdot\psi}\right].$$

Proof.

$$\begin{aligned}\overline{\mathcal{D}}\,\overline{\psi} &= \sum_i \overline{e}_i\,\bar{\cdot}\,\overline{\nabla}_{\overline{e}_i}\overline{\psi} = e^{-u}\sum_i \overline{e}_i\,\bar{\cdot}\,\overline{\nabla}_{e_i}\overline{\psi}\\ &= e^{-u}\sum_i \overline{e}_i\,\bar{\cdot}\,\left[\,\overline{\nabla_{e_i}\psi} - \frac{1}{2}\overline{e_i\cdot\mathrm{grad}(u)\cdot\psi} - \frac{1}{2}e_i(u)\overline{\psi}\,\right]\\ &= e^{-u}\left[\,\overline{\mathcal{D}\psi} + \frac{n-1}{2}\,\overline{\mathrm{d}u\;\cdot\psi}\,\right]\end{aligned}$$

□

Proof of Proposition 5.13. Let $f := e^{-\frac{n-1}{2}u}$. Then

$$\begin{aligned}\overline{\mathcal{D}}(f\,\overline{\psi}) &= f\,\overline{\mathcal{D}}\,\overline{\psi} + e^{-u}\,\overline{\mathrm{d}f\cdot\psi}\\ &= fe^{-u}\,\overline{\mathcal{D}\psi} + fe^{-u}\,\frac{n-1}{2}\,\overline{\mathrm{d}u\cdot\psi} + e^{-u}\,\overline{\mathrm{d}f\cdot\psi}\\ &= fe^{-u}\,\overline{\mathcal{D}\psi},\end{aligned}$$

since $\mathrm{d}f = -\frac{n-1}{2}\,\mathrm{d}u$. □

Theorem 5.18. [Hi1] *Let $(M^n,\, g)$ be a compact Riemannian manifold and $\mathcal{D}\psi = \lambda\psi$ for a non-trivial spinorfield ψ. Then*

$$\lambda^2 \geq \frac{n}{4(n-1)}\sup_u \inf_M \left(\overline{S}\; e^{2u}\right). \tag{5.9}$$

In case of equality, ψ is a Killing spinor.

Proof. The Schrödinger-Lichnerowicz formula and $|\nabla\psi|^2 = \frac{1}{n}|\mathcal{D}\psi|^2 + |\,\mathcal{P}\,\psi|^2$ yield

$$\begin{aligned}0 \leq \int_M |\,\mathcal{P}\,\psi|^2\;\nu_g &= \int_M \left(|\nabla\psi|^2 - \frac{1}{n}|\mathcal{D}\psi|^2\right)\;\nu_g\\ &= \int_M \left(\frac{n-1}{n}|\mathcal{D}\psi|^2 - \frac{1}{4}S|\psi|^2\right)\;\nu_g\end{aligned}$$

for any section $\psi \in \Gamma(\Sigma M)$.

For a fixed conformal change $u\colon M \to \mathbb{R}$ and $\overline{\varphi} := e^{-\frac{n-1}{2}u}\;\overline{\psi}$ we get

$$0 \leq \int_M |\overline{\mathcal{P}}\;\overline{\varphi}|^2\;\nu_{\overline{g}} = \int_M \left(\frac{n-1}{n}|\overline{\mathcal{D}}\;\overline{\varphi}|^2 - \frac{1}{4}\,\overline{S}|\overline{\varphi}|^2\right)\;\nu_{\overline{g}}.$$

By Proposition 5.13 we get

$$\overline{\mathcal{D}}\,\overline{\varphi} = \lambda e^{-u}\overline{\varphi}\,.$$

So

$$\begin{aligned}0 &\leq \frac{n-1}{n}\int_M \left(\lambda^2 - \frac{1}{4}\frac{n}{n-1}\overline{S}\;e^{2u}\right)|\overline{\psi}|^2 e^{-2u}\nu_{\overline{g}}\\ 0 &\leq \frac{n-1}{n}\left(\lambda^2 - \frac{1}{4}\frac{n}{n-1}\inf_M \overline{S}\;e^{2u}\right)\int_M |\overline{\psi}|^2 e^{-2u}\nu_{\,\overline{g}}.\end{aligned}$$

Hence $\lambda^2 \geq \frac{1}{4}\frac{n}{n-1}\inf_M\left(\overline{S}\;e^{2u}\right)$. So inequality (5.9) holds.

In the case of equality in (5.9) we have $\overline{\mathcal{D}}\,\overline{\varphi} = \lambda e^{-u}\overline{\varphi}$ and $\overline{\mathcal{P}}\;\overline{\varphi} = 0$. Thus

$$\overline{\nabla}_{\overline{X}}\,\overline{\varphi} + \frac{\lambda e^{-u}}{n}\;\overline{X}\,\overline{\cdot}\,\overline{\varphi} = 0\,, \qquad \forall X \in \Gamma(TM)\,,$$

which by (5.12) applied, w.r.t the metric $\overline{g}$, to $\overline{\varphi}$ with $f = \lambda e^{-u}$ implies that u is constant, hence ψ is a Killing spinor. □

Definition 5.19 (Yamabe operator)**.** The *Yamabe operator* of a manifold of dimension $n \geq 3$ is defined by (see [Be], for example):

$$\mathrm{L} = 4\,\frac{n-1}{n-2}\,\Delta + S.$$

This operator is related to the Yamabe problem: Is there a metric of constant scalar curvature in a given class of conformal Riemannian metrics on a compact manifold?

Corollary 5.20. *Any eigenvalue λ of the Dirac operator satifies:*

a) *For $n \geq 3$,*

$$\lambda^2 \geq \frac{n}{4(n-1)}\,\mu_1, \quad n \geq 3,$$

where μ_1 stands for the first eigenvector of the scalar conformal Laplacian L *(see* [Hi1]*).*

b) *For $n = 2$,*

$$\lambda^2 \geq \frac{2\pi\,\chi(M^2)}{\mathrm{Area}(M^2, g)}, \quad n = 2.$$

where $\chi(M^2)$ is the Euler characteristic class (see [Bä1]*).*

In case of equality in a) or b) the eigenspinors associated with λ are Killing spinors.

Proof. a) Let $u\colon M \to \mathbb{R}$ be a conformal change. Then the scalar curvature transforms according to

$$\overline{S}e^{2u} = S + 2(n-1)\Delta u - (n-1)(n-2)|\mathrm{d}u|^2. \tag{5.10}$$

For $n \geq 3$, define the function h by

$$h^{\frac{4}{n-2}} = e^{2u}\,. \tag{5.11}$$

Then Equation (5.10) simplifies to

$$\overline{S}h^{\frac{4}{n-2}} = h^{-1}\,\mathrm{L}\,h.$$

One knows from general theory, that an eigenfunction h_1 to the first eigenvalue μ_1 of L can be chosen to be positive. Moreover, it is known that μ_1 is positive iff, in the conformal class, there is a metric of positive scalar curvature. Thus $h_1 > 0$ and

$$h_1^{-1}\,\mathrm{L}\,h_1 = \mu_1 > 0\,.$$

Now, take u_1 associated to h_1 by (5.11). Then

$$\overline{S}\,e^{2u_1} = \overline{S}\,h_1^{\frac{4}{n-2}} = h_1^{-1}\,\mathrm{L}\,h_1 = \mu_1.$$

It can be shown [Hi3], that if

$$\overline{S}e^{2u_1} = const.,$$

then $\overline{S}e^{2u_1} = \sup_u \inf_M(\overline{S}e^{2u})$.

b) For $n = 2$, the transformation of the scalar curvature under conformal change of the metric, gives

$$\overline{S}e^{2u} = S + 2\Delta u.$$

Now

$$\begin{aligned}\inf_M(\overline{S}e^{2u}) &\leq \frac{1}{\mathrm{Area}(M^2,g)}\int_M \overline{S}e^{2u}\nu_g = \frac{1}{\mathrm{Area}(M^2,g)}\int_M (S+2\Delta u)\nu_g \\ &\leq \frac{1}{\mathrm{Area}(M^2,g)}\int_M S\,\nu_g = \frac{4\pi\,\chi(M^2)}{\mathrm{Area}(M^2,g)}.\end{aligned}$$

by Stokes and Gauß–Bonnet theorem.

Let u_1 be a solution of

$$2\Delta u = \frac{1}{\mathrm{Area}(M^2,g)}\int_M S\nu_g - S.$$

Then we finally get

$$\overline{S}e^{2u_1} = S + 2\Delta u_1 = \frac{1}{\mathrm{Area}(M^2,g)}\int_M S\nu_g = \frac{4\pi\,\chi(M^2)}{\mathrm{Area}(M^2,g)}.$$

□

Remark 5.21. With the help of the conformal covariance of the Dirac operator and of the Sobolev embedding theorem for pseudo-differential operators, J. Lott [Lo] showed the existence of a qualitative conformal lower bound for the eigenvalues of the Dirac operator. Moreover, for further formulation of (5.9) see [Hi3].

Theorem 5.22. [Hi1, Li2, Li3]

a) *If ψ is a Killing spinor and $\alpha \in \Omega^k(M)$ an arbitrary harmonic form of degree $k \neq 0$, n. Then*

$$\alpha \cdot \psi = 0. \tag{5.12}$$

b) *If M admits a non–trivial Killing spinor, then there exists no non–trivial parallel form $\alpha \in \Omega^k(M)$, $k \neq 0$, n.*

Proof. a) The Killing spinor ψ is an eigenvector to the smallest eigenvalue λ_1 of the Dirac operator $\mathcal{D}$. Using now the formulas

$$\mathrm{d} = \sum e_i \wedge \nabla_{e_i}, \tag{5.13}$$

$$\delta = -\sum e_i \lrcorner \nabla_{e_i}, \tag{5.14}$$

By Proposition 2.8 and

$$\sum_i e_i \cdot \alpha \cdot e_i = (-1)^{k-1}(n-2k)\alpha, \tag{5.15}$$

we get

$$\begin{aligned}\mathcal{D}(\alpha\cdot\psi) &= \sum_i e_i\cdot\nabla_{e_i}(\alpha\cdot\psi) = \sum_i e_i\cdot(\nabla_{e_i}\alpha\cdot\psi + \alpha\cdot\nabla_{e_i}\psi)\\ &= \Big((\mathrm{d}+\delta)\alpha\Big)\cdot\psi + \sum_i e_i\cdot\alpha(-\frac{\lambda_1}{n}e_i\cdot\psi)\\ &= \Big(\underbrace{(\mathrm{d}+\delta)\alpha}_{=0}\Big)\cdot\psi + \underbrace{(-1)^k\lambda_1(1-\frac{2k}{n})}_{:=\lambda}\ \underbrace{\alpha\cdot\psi}_{:=\varphi},\end{aligned}$$

since on compact manifolds $\Delta\alpha = 0 \iff \mathrm{d}\alpha = \delta\alpha = 0$. Now $|\lambda| < |\lambda_1|$ implies by Friedrich's inequality (Theorem 5.3)

$$\alpha\cdot\psi = 0.$$

b) Let α be a parallel k–form. Then by (5.12), $\alpha\cdot\psi = 0$. Now for any vector field X,

$$0 = \nabla_X(\alpha\cdot\psi) = \Big(\underbrace{\nabla_X\alpha}_{=0}\Big)\cdot\psi + \alpha\cdot\nabla_X\psi.$$

Since ψ is a Killing spinor, it follows that

$$\alpha\cdot X\cdot\psi = 0\,.$$

On the other hand, by Proposition 2.8 it follows that

$$\alpha\cdot X = (-1)^k\Big(X\cdot\alpha + 2X\lrcorner\,\alpha\Big)\,,$$

hence for any vector field X, one has

$$(X\lrcorner\,\alpha)\cdot\psi = 0\,,$$

which by induction on k, shows that $\psi = 0$.

□

This theorem has immediate important topological consequences. For example, if there is a Killing spinor on a compact manifold, then the first de Rham cohomology is zero, $H^1_{\mathrm{dR}}(M,\mathbb{R}) = 0$. Since the Kähler form $\Omega(X,Y) = g(X,JX)$ is parallel, there are no Killing spinors on compact Kähler manifolds.

5.3. **Holonomy and Eigenvalues of the Dirac Operator.** We have seen that on a compact Riemannian spin manifold $(M^n,\, g)$, any eigenvalue λ of $\mathcal{D}$ satisfies Friedrich's inequality, $\lambda^2 \geq \frac{1}{4}\frac{n}{n-1}S_0$, (see Theorem 5.3). The key argument wrt Theorem 5.1 ii), is to replace $|\nabla\psi|^2$ in the Schrödinger-Lichnerowicz formula by

$$|\nabla\psi|^2 = \frac{1}{n}|\mathcal{D}\psi|^2 + |\,\mathcal{P}\,\psi|^2$$

and then observe that $|\,\mathcal{P}\,\psi|^2 \geq 0$ translates to Theorem 5.3. If $\lambda^2 = \frac{1}{4}\frac{n}{n-1}S_0$, then for an eigenspinor ψ associated with λ, it follows that $\mathcal{P}\,\psi = 0$. Therefore ψ is a Killing spinor, i.e., it satifies the overdetermined system

$$\forall X \in \Gamma(TM), \qquad \nabla_X \psi \pm \frac{1}{2} X \cdot \psi = 0\,, \tag{5.16}$$

where we normalized the metric so that $S_0 = n(n-1)$.

S. Sulanke (see [BFGK]) computed the spectrum of the the Dirac operator on the round sphere. Notice that, applying (5.16) with $X = e_i$, taking its Clifford multiplication with e_i and summing over i, yields to $\mathcal{D}\psi = \pm\frac{n}{2}\,\psi$. It turns out that the n-sphere has a maximal number of Killing spinors. This could also be seen by using a spinorial Gauss-type formula for $S^n \subset \mathbb{R}^{n+1}$ [Tr, Bä3].

For the classification of maniflods with Killing spinors one has to distinguish between even and odd dimensions. Using the twistor construction, Friedrich proved that in 4-dimensions, only the round sphere has Killing spinors (see [Fr1]). The same result holds on 8-dimensional manifolds [Hi2]. The proof relies on the triality principle and on the Obata-Lichnerowicz theorem. In dimension 6, using the twistor construction on S^4 and $\mathbb{C}P^2$, Grunewald proved that nearly-Kähler but non-Kähler manifolds [BFGK] carry such spinors.

In odd dimensions, this question has been systematically studied by T. Friedrich and I. Kath [BFGK]. Namely, in dimensions 5 and 7, there is a one-to-one correspondence between Killing spinors and Einstein-Sasaki structures. It turns out that such charaterizations depend on the number of solutions of (5.16), i.e., the number of Killing spinors.

In 1989, using the Berger-Simon classification theorem, M. Wang classified simply connected, complete, irreducible manifolds with parallel spinors. With the help of this classification result and a classical argument already used by S. Gallot [Ga1], C. Bär [Bä2] gave a geometrical description of simply connected complete manifolds with Killing spinors. More precisely, denote such a manifold by (M^n, g_M) and consider the manifold (N^{n+1}, g_N) where $g_N := r^2 g_M \oplus dr^2$ is the warped product metric on $N^{n+1} := M \underset{r^2}{\times} \mathbb{R}^*_+$, then if ∇^N denotes the Levi-Civita connection of (N^{n+1}, g_N), for $\psi \in \Gamma(\Sigma N)$, one gets

$$\nabla^N \psi \equiv 0 \Longleftrightarrow \psi\big|_M \quad \text{is a Killing spinor on } (M^n, g_M)\,.$$

Note that, if $(M^n, g_M) = (S^n, \mathrm{S}tandard)$, then the warped product metric g_N is the standard metric on the euclidean space

$$\mathbb{R}^{n+1} \setminus \{0\} = S^n \underset{r^2}{\times} \mathbb{R}^*_+\,.$$

In this characterization dimensions 6 and 7 are exceptional and lead to the construction of incomplete Riemannian manifolds with holonomy G_2 and Spin_7. For example, if the dimension n is even and $n \neq 6$, then the spheres are the only such manifolds. It should be pointed out that, supergravity and superstring theories concern these two exceptional cases.

We have seen that on compact Kähler manifolds, there are no Killing spinors. For such manifolds, K. D. Kirchberg [Ki1] proved the following. For any eigenvalue λ of $\mathcal{D}$, if $n = 0 \mod 4$ (even complex dimension), then

$$\lambda^2 \geq \frac{1}{4}\frac{n}{n-2}\, S_0\,. \tag{5.17}$$

If $n = 2 \mod 4$ (odd complex dimension), then

$$\lambda^2 \geq \frac{1}{4}\frac{n+2}{n}\, S_0\,. \tag{5.18}$$

The proof of Kirchberg's inequalities is based on the double decomposition of the covariant derivative of an eigenspinor. First, that of the spinor bundle under the action, by Clifford multiplication, of the parallel Kähler 2-form, second, as in the Riemannian case, the decomposition wrt the image and the kernel of Clifford multiplication. This lead to the notion of Kählarian twistor operators. Eigenspinors which are in the kernel of the Kählarian twistor operators are then called Kählerian Killing spinors. Manifolds with such spinors are called *limiting Kählerian manifolds*.

All these manifolds have been recently geometrically described. In low dimensions, there are classification results due to Kirchberg [Ki3] in dimension 6, and to Th. Friedrich [Fr3] in dimension 4. In odd complex dimension, A. Moroianu [Mo1] showed that a limiting Kähler manifold is characterized as the twistor space asociated to a Quaternion-Kähler manifold of positive scalar curvature.

For limiting Kählerian manifold of even complex dimension [Li5, Mo3], it turns out that the universal cover of such a manifold is isometric to a Riemannian product $M \times \mathbb{R}^2$, where M is a limiting Kählerian manifold of odd complex dimension.

The third interesting case in this setup is the family of compact spin manifolds with positive scalar curvature and holonomy $\mathrm{Sp}_n \cdot \mathrm{Sp}_1$. These manifolds are characterized by the existence of a parallel 4-form. Such manifolds are necessary Einstein of dimension $n = 0 \mod 4$. First J.-L. Milhorat [Mi1] computed the spectrum of the Dirac operator on the model space, $\mathbb{H}P^n$. Then, in [HM1, HM2, HM3] it is proved that for $n = 8, 12$, any eigenvalue of the Dirac operator satisfies

$$\lambda^2 \geq \frac{1}{4}\frac{n+12}{n+8} S_0\,. \tag{5.19}$$

The approach used is similar to that in the Kähler situation but the argument was not sufficient to prove (5.19) in higher dimensions. In fact, it turned out that, in higher dimensions, some additional terms were missing in the definition of the Quaternion-Kähler twistor operator introduced in [HM3]. Note that, in [BH] where extensions of (5.9) are given, it is also shown how one can make use of representation theory to systematically get optimal Bochner-Weitzenböck type formulae for natural differential operators.

In [KSW1, KSW2, KSW3], it is proved that (5.19) is true in all dimensions and that the only limiting manifold is the Quaternion-Kähler projective space. For this, the authors use representation theory to define natural twistor operators based on a double decomposition under the action of Sp_n and Sp_1. Recently, with the help of represntation theory, J.-L. Milhorat [Mi2] gave a simple proof of (5.19).

Summing up, we have seen that the optimal lower eigenvalue estimates depend on the holonomy of the manifold. For manifolds with holonomy SO_n, U_m or $\mathrm{Sp}_1 \cdot \mathrm{Sp}_n$, i.e. Riemannian, Kähler or Quaternion-Kähler manifolds, the optimal lower bounds for the eigenvalues of the square of the Dirac operator are

$$\underbrace{\frac{n}{n-1}\frac{S_0}{4}}_{\text{Riemannian}} \leq \underbrace{\frac{n+2}{n}\frac{S_0}{4}}_{\text{Kähler, } n=2 \mod 4} \leq \underbrace{\frac{n}{n-2}\frac{S_0}{4}}_{\text{Kähler, } n=0 \mod 4} \leq \underbrace{\frac{n+12}{n+8}\frac{S_0}{4}}_{\text{Quaternion-Kähler}}.$$

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