BASIC DIFFERENTIAL GEOMETRY: SEMI-RIEMANNIAN METRICS

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Introduction

I discuss basic features of semi-Riemannian manifolds: Levi-Civita connection and curvature tensor, sectional curvature, Ricci and scalar curvature, and, as examples, some spaces of constant curvature.

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Conventions

If $U \subset \mathbb{R}^m$ is open, V is a real (or complex) vector space (of finite dimension), and $\varphi : U \to V$ is a smooth function, then the partial derivative of φ with respect to x_i is denoted in the following different ways,

$$\varphi_i = \varphi_{x_i} = \frac{\partial \varphi}{\partial x^i} = d\varphi \cdot \frac{\partial}{\partial x^i}.$$

Analogous notation will be used for higher partial derivatives. There are other objects with indices, where the indices have a different meaning. But it seems that there is no danger of confusion.

Let M be a manifold. By $\mathcal{F}(M)$ and $\mathcal{V}(M)$ we denote the spaces of smooth real valued functions and smooth vector fields on M, respectively. Recall that tangent vectors of M act as derivations on smooth maps with values in vector spaces, $\varphi : M \to V$. For $X \in \mathcal{V}(M)$, we use the notations $Xf = df \cdot X$ for the induced smooth function $M \ni p \mapsto X(p)(f) \in V$.

A frame of TM over a subset U of M consists of a tuple $\Phi = (X_1, \dots, X_m)$ of smooth vector fields of M over U such that $(X_1(p), \dots, X_m(p))$ is a basis of T_pM, for all $p \in U$. If X is a vector field of M over U, then the map $\xi : U \to \mathbb{R}^m$ with $X = \xi^i X_i$ is called the *principal part* of X with respect to Φ. In the last formula, the *Einstein convention* is in force. I will use it throughout: If in a term an index occurs as upper and lower index, then it is understood that the sum over that index is taken.

If U is open, Φ is a frame of TM over U, and X is a smooth vector field of M over U, then the principal part ξ of X is smooth. If $x : U \to U'$ a coordinate chart of M, then

$$(X_1, \dots, X_m) := \Big(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}\Big) \tag{0.1}$$

is a frame of TM over U. We call it the *frame associated to x*. For this frame, the principal part of a vector field X of M over U is given by $dx \cdot X$.

1. Semi-Riemannian metrics

A *semi-Riemannian metric* on a manifold M is a family g of non-degenerate symmetric bilinear forms

$$g_p : T_pM \times T_pM \to \mathbb{R}, \quad p \in M, \tag{1.1}$$

such that the function $g(X,Y) : p \mapsto g_p(X(p), Y(p))$ is smooth for all smooth vector fields X, Y on M. A *semi-Riemannian manifold* is a manifold M together with a semi-Riemannian metric. Semi-Riemannian metrics and manifolds will also be called SR-metrics and SR-manifolds. If g_p is positive definite for all p in M, then g is called a *Riemannian metric* on M and M a *Riemannian manifold.* If M has dimension at least 2 and g_p has index 1 for all p in M, then g is called a *Lorentz metric* and M a *Lorentz manifold.*

In what follows, M is an SR-manifold with an SR-metric g. For $p \in M$, the form g_p is mostly denoted $\langle \cdot, \cdot \rangle_p$ or simply $\langle \cdot, \cdot \rangle$, respectively, if there is no danger of confusion.

Let $U \subset M$ be open and $\Phi = (X_1, \dots, X_m)$ be a smooth frame of TM over U. Then the entries of the *fundamental matrix of* g with respect to Φ are

$$g_{ij}(p) := \langle X_i(p), X_j(p) \rangle_p, \quad p \in U. \tag{1.2}$$

The functions $g_{ij} : U \to \mathbb{R}$ are smooth. Strictly speaking, we only required in the above definition that the functions $g(X,Y)$ are smooth for all smooth vector fields X, Y on M. However, smoothness is a local property, hence we can apply Lemma B.1.3 to conclude that the functions g_{ij} are smooth as well.

For vector fields X, Y on U with principal parts ξ and η, that is, $X = \xi^i X_i$ and $Y = \eta^i X_i$, we obtain

$$\langle X, Y \rangle = g_{ij} \xi^i \eta^j. \tag{1.3}$$

where we suppress, for better readability, the dependence on $p \in U$. If Φ is the frame associated to a coordinate chart (x, U) of M, then $(dx^1, \dots, dx^m)$ is the corresponding dual frame of T^*M over U, and then we can rewrite (1.3) as

$$g = g_{ij} dx^i dx^j. \tag{1.4}$$

Since g is non-degenerate, the matrix (g_{ij}) is symmetric with non-vanishing determinant. In the Riemannian case, (g_{ij}) is positive definite. Since (g_{ij}) is symmetric, the same holds for the inverse matrix, denoted (g^{ij}). Recalling the formula for the inverse of a matrix, we see that the functions g^{ij} are smooth on U. The meaning of the inverse matrix is explained below.

Examples 1.1. 1) *The constant metric*: Let V be a real vector space of dimension m and $\langle \cdot, \cdot \rangle$ be a non-degenerate symmetric bilinear form on V. We use the canonical identification of T_pV with V: Recall that $v \in V$ is identified with $c'(0)$, where $c = c(t) = p + tv$. For $v, w \in T_pV \cong V$, we set

$$g_p(v, w) = \langle v, w \rangle, \tag{1.5}$$

where v, w are considered as tangent vectors at p on the left hand side and as elements of V on the right. With respect to the above identification of tangent spaces with V, a vector field X on V is a smooth map $X : V \to V$. For vector fields X, Y on V we have

$$g_p(X(p), Y(p)) = \langle X(p), Y(p) \rangle$$

and this is a smooth function of p. Now $\langle \cdot, \cdot \rangle$ is non-degenerate, hence g is a PR-metric on V.

2) *Riemann's metric*: Let V and $\langle \cdot, \cdot \rangle$ be as in Example 1) above and let $\kappa \neq 0$ be a constant. On the open set

$$V_\kappa = \{p \in V \mid \kappa \langle p, p \rangle \neq -1\}$$

we modify the metric from Example 1 as follows:

$$g_p(v, w) = \frac{4\langle v, w \rangle}{(1 + \kappa \langle p, p \rangle)^2}, \tag{1.6}$$

that is, $g_p = f_\kappa \cdot \langle \cdot, \cdot \rangle$, where $f_\kappa : V_\kappa \to \mathbb{R}$ is the smooth function

$$f_\kappa(p) = \frac{4}{(1 + \kappa \langle p, p \rangle)^2}.$$

This metric appears, in a slightly different form, in Riemann's Inaugural Lecture "On the hypotheses which lie at the Foundations of Geometry". It is a model for a space of constant sectional curvature, see [IS] in connection with Example 1.3.2 below.

3) Let V be a vector space of dimension $m + 1$ and $\langle \cdot, \cdot \rangle$ be a non-degenerate symmetric bilinear form on V. For $\alpha \neq 0$, set

$$Q_\alpha = \{p \in V \mid \langle p, p \rangle = \alpha\} \tag{1.7}$$

and assume $Q_\alpha \neq \emptyset$. Then Q_α is a smooth submanifold of V of dimension m, a level set of the function $f(p) = \langle p, p \rangle$ on V.

If V is Euclidean space, that is, if $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^{m+1}$, and if $\alpha > 0$, then Q_α is the sphere $S^m(r)$ of radius $r = \sqrt{\alpha}$ about 0. If V is Minkowski space, $V = \mathbb{R}^{m+1}_1$, then Q_{-1} consists of two connected components. The component $H^m = \{p \in Q_{-1} \mid p^0 > 0\}$ is called m-dimensional (real) *hyperbolic space.*

The tangent space of Q_α in a point p is given by the kernel of $df(p)$ for the above function f,

$$T_pQ_\alpha \cong \{v \in V \mid df|_p(v) = 0\} = \{v \in V \mid \langle p, v \rangle = 0\} =: p^N.$$

We obtain a semi-Riemannian metric on Q_α by restriction of the semi-Riemannian metric in Example 1) above: for $v, w \in T_pQ_\alpha$ we set $g_p(v, w) = \langle v, w \rangle$. For $p \in Q_\alpha$, the restriction of $\langle \cdot, \cdot \rangle$ to p^N is non-degenerate since $\langle p, p \rangle = \alpha \neq 0$. Hence g_p is a non-degenerate symmetric bilinear form on $T_pQ_\alpha \cong p^N$. With respect to the identifications $T_pQ_\alpha \cong p^N$, a vector field on Q_α is a smooth map $X : Q_\alpha \to V$

with $X(p) \in p^N$ for all $p \in Q_\alpha$. It follows that $\langle X, Y \rangle$ is smooth for any two smooth vector fields X and Y on Q_α, hence Q_α together with the above family (g_p) of non-degenerate bilinear forms is a semi-Riemannian manifold.

1.1. **Isometries.** All the data needed in the definition of an SR-manifold is a manifold M together with an SR-metric g on M. Thus two SR-manifolds M and N are indistinguishable from the point of view of SR-geometry if there is a diffeomorphism $f : M \to N$ preserving the metric, that is, f satisfies

$$\langle f_{*p}v, f_{*p}w\rangle_{f(p)} = \langle v, w\rangle_p \quad \text{for all } p \in M \text{ and } v, w \in T_pM. \tag{1.8}$$

Diffeomorphisms of this kind are called *isometries.* Clearly, the identity map $\mathrm{id} : M \to M$, the inverse of an isometry, and the composition of isometries are isometries. In particular, the set $I(M)$ of isometries of M, that is, of isometries $f : M \to M$, is a group, the *isometry group* of M.

A map $f : M \to N$ between SR-manifolds M and N *of the same dimension* is called a *local isometry* if M and N have the same dimension and if f satisfies (1.8). Isometries are local isometries. The composition of local isometries is a local isometry. Note that a local isometry is of rank $\dim M = \dim N$ everywhere since (1.8) implies that f_{*p} does not have a kernel. Hence local isometries are local diffeomorphisms.

Proposition 1.2. *Let (M, g) and (N, h) be SR-manifolds of the same dimension, and $f : M \to N$ be a smooth map. Then f is a local isometry if and only if one of the following two equivalent conditions holds:*

1) For all coordinate charts (x, U) of M and (y, V) of N we have, on $U \cap f^{-1}(V)$,

$$g_{ij} = h_{kl}\, f_i^k f_j^l, \quad \textit{where } f^i := y^i \circ f,\ 1 \le i \le m.$$

2) For every $p \in M$, there are coordinate charts (x, U) about p in M and (y, V) about $f(p)$ in N such that $x = y \circ f$ and $g_{ij} = h_{ij} \circ f$. □

Example 1.3. 1) Consider the quadric $Q_\alpha = \{p \in V \mid \langle p, p\rangle = \alpha\}$ as in Example 1.1.3. If $L : V \to V$ is a linear isomorphism preserving $\langle\cdot,\cdot\rangle$, then $L(Q_\alpha) \subset Q_\alpha$, and hence the restriction $L : Q_\alpha \to Q_\alpha$ is an isometry. It follows that $\mathrm{Aut}(V, \langle\cdot,\cdot\rangle) \subset I(Q_\alpha)$. We will see in [IS] that actually $\mathrm{Aut}(V, \langle\cdot,\cdot\rangle) = I(Q_\alpha)$.

2) Consider Examples 1.1.2 and 1.1.3 above. Let p_0 be a point in Q_α. Let $W = p_0^N$. Any $p \in V$ can be written as $p = xp_0 + q$ with $x \in \mathbb{R}$ and $q \in W$, where $x = x(p)$ and $q = q(p)$ are linear in $p \in V$. On the complement U_0 of the hyperplane $x(p) = 1$ we have the *stereographic projection*

$$h : U_0 \to W\,, \quad h(p) = \frac{q}{1-x}. \tag{1.9}$$

As usual we identify tangent vectors of V with vectors in V. With respect to the above decomposition of V, we write tangent vectors of V as $v = \xi p_0 + w$ with $w \in W$. Then

$$dh(p) \cdot (\xi p_0 + w) = \frac{\xi}{(1-x)^2} q + \frac{1}{(1-x)} w.$$

For $p \in Q_\alpha \cap U_0$ we have

$$\alpha = \langle p, p\rangle = \langle xp_0 + q, xp_0 + q\rangle = \alpha x^2 + \langle q, q\rangle.$$

Let $v_1 = \xi_1 p_0 + w_1$ and $v_2 = \xi_2 p_0 + w_2$ be tangent to Q_α at p. Then

$$0 = \langle p, \xi_i p_0 + w_i\rangle = \alpha x \xi_i + \langle q, w_i\rangle, \quad i = 1, 2.$$

Therefore

$$\begin{aligned}
&\langle dh(p)\cdot v_1, dh(p)\cdot v_2\rangle \\
&\qquad = \frac{1}{(1-x)^2}\left(\frac{\xi_1\xi_2\langle q,q\rangle}{(1-x)^2} + \frac{\xi_1\langle q,w_2\rangle + \xi_2\langle q,w_1\rangle}{1-x} + \langle w_1,w_2\rangle\right) \\
&\qquad = \frac{1}{(1-x)^2}\left(\frac{\xi_1\xi_2\alpha(1+x)}{1-x} - \frac{2\xi_1\xi_2\alpha x}{1-x} + \langle w_1,w_2\rangle\right) \\
&\qquad = \frac{1}{(1-x)^2}\big(\xi_1\xi_2\alpha + \langle w_1,w_2\rangle\big) = \frac{1}{(1-x)^2}\cdot\langle v_1,v_2\rangle.
\end{aligned}$$

Now let $\kappa = 1/\alpha$. Then we have, for $p \in Q_\alpha$,

$$\begin{aligned}
\frac{2}{1+\kappa\,\langle h(p),h(p)\rangle} &= \frac{2\alpha}{\alpha + \frac{\langle q,q\rangle}{(1-x)^2}} = \frac{2\alpha(1-x)^2}{\alpha(1-x)^2 + \langle q,q\rangle} \\
&= \frac{2\alpha(1-x)^2}{\alpha(1-x)^2 + \alpha(1-x^2)} = 1 - x,
\end{aligned}$$

and hence $dh(p) : T_pQ_\alpha \to T_{h(p)}W \cong W$ is an orthogonal transformation if we equip $T_{h(p)}W$ with the metric from Example 2. It follows that

$$h : Q_\alpha \cap U_0 \to W_\kappa := \{u \in W \mid \kappa\langle u,u\rangle \neq -1\}$$

is an isometry, where W_κ is equipped with Riemann's metric.

1.2. **Metrics and duality.** If B is a non-degenerate bilinear form on a vector space V of finite dimension, then the map $V \to V^*$, $v \mapsto B(v,\cdot)$, is an isomorphism. In the case of an SR-manifold, this leads to a correspondence between 1-forms and vector fields.

LEMMA 1.4. *There is a one-to-one correspondence between 1-forms and vector fields on M, where a 1-form ω corresponds to a vector field X iff*

$$\omega(Z) = \langle X, Z\rangle \quad \text{for all } Z \in \mathcal{V}(M).$$

With respect to a local coordinate chart (x,U) of M this means

$$\xi^i = g^{ij}\omega_j \quad \text{respectively} \quad \omega_j = g_{ji}\xi^i,$$

where $(\xi^1,\ldots,\xi^m)$ and $(\omega_1,\ldots,\omega_m)$ are the corresponding principal parts of X and ω with respect to x, respectively.

Proof. Since $\langle \cdot, \cdot \rangle_p$ is a non-degenerate symmetric bilinear form on T_pM, $p \in M$, the map $T_pM \to T_p^*M$, $v \mapsto \langle v, . \rangle_p$, is an isomorphism. Hence the correspondence between vector fields X and 1-forms $\omega = \langle X, . \rangle$ is one-to-one.

Let (x, U) be a coordinate chart of M. Let ξ be the corresponding principal part of X. Then

$$\langle X, X_k \rangle = \xi^i g_{ik} = \xi^i g_{ij} dx^j(X_k), \quad 1 \leq k \leq m,$$

and hence $\omega = g_{ij}\xi^i dx^j$ over U. □

2. The Levi-Civita connection

For the general theory of connections we refer to [CG] and [VC].

Let M be an SR-manifold. We say that a connection on M is *metric* if it satisfies the product rule,

$$X\langle Y, Z\rangle = \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle \tag{2.1}$$

for all $X, Y, Z \in \mathcal{V}(M)$.

Fundamental Lemma 2.1. *There is a unique metric and symmetric connection D on M, the* Levi-Civita connection *D of M.*

The Levi-Civita connection is determined by the *Koszul formula*

$$\begin{aligned} 2\langle D_X Y, Z\rangle =& X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle \\ & -\langle X, [Y, Z]\rangle - \langle Y, [X, Z]\rangle + \langle Z, [X, Y]\rangle. \end{aligned} \tag{2.2}$$

Proof of Lemma 2.1. We derive necessary conditions on D: Suppose that D is a metric and symmetric connection on M. Then, for all $X, Y, Z \in \mathcal{V}(M)$,

$$\begin{aligned} X\langle Y, Z\rangle &= \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle \\ Y\langle X, Z\rangle &= \langle D_Y X, Z\rangle + \langle X, D_Y Z\rangle \\ Z\langle X, Y\rangle &= \langle D_Z X, Y\rangle + \langle X, D_Z Y\rangle. \end{aligned}$$

Therefore,

$$\begin{aligned} &X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle \\ &\qquad = \langle D_X Y, Z\rangle + \langle D_Y X, Z\rangle + \langle Y, D_X Z - D_Z X\rangle + \langle X, D_Y Z - D_Z Y\rangle. \end{aligned}$$

Since D is symmetric, the right hand side is equal to

$$2\langle D_X Y, Z\rangle - \langle [X, Y], Z\rangle + \langle Y, [X, Z]\rangle + \langle X, [Y, Z]\rangle\,.$$

We conclude that D satisfies the Koszul formula.

Vice versa, we show now that the Koszul formula determines a metric and symmetric connection. For fixed smooth vector fields X and Y, consider the right hand side RHS of the Koszul formula as a map depending on Z, RHS $= \omega(Z) \in \mathcal{F}(M)$. It is easy to see that ω is tensorial in Z. Hence ω is a 1-form.

By Lemma 1.4, there is a unique vector field, which we call $2D_X Y$, such that $\omega(Z) = \langle 2D_X Y, Z\rangle$ for all vector fields Z. We obtain a map

$$D : \mathcal{V}(M) \times \mathcal{V}(M) \to \mathcal{V}(M).$$

Clearly the right hand side of the Koszul formula (and hence D) is additive in X and Y. We also have

$$\begin{aligned}2\langle D_X(\varphi Y), Z\rangle &= X(\varphi)\cdot\langle Y,Z\rangle + \varphi\cdot X\langle Y,Z\rangle + \varphi\cdot Y\langle X,Z\rangle \\ &\quad - Z(\varphi)\cdot\langle X,Y\rangle - \varphi\cdot Z\langle X,Y\rangle - \langle X, \varphi\cdot[Y,Z] - Z(\varphi)\cdot Y\rangle \\ &\quad - \varphi\cdot\langle Y,[X,Z]\rangle + \langle Z, \varphi\cdot[X,Y] + X(\varphi)\cdot Y\rangle \\ &= 2\langle X(\varphi)\cdot Y, Z\rangle + 2\langle \varphi\cdot D_X Y, Z\rangle,\end{aligned}$$

and hence D is a derivation in Y. An easier calculation shows that D is $\mathcal{F}(M)$-homogeneous in the variable X. Hence D is a connection. Using the Koszul formula we see that D is metric and symmetric. □

By our general discussion about connections, the Koszul formula also applies to vector fields on open subsets of M.

Let $(X_1, \ldots, X_m)$ be a frame of TM over an open subset $U \subset M$. Then the *Christoffel symbols* are the coefficient functions $\Gamma_{ij}^k : U \to \mathbb{R}$ defined by

$$D_{X_i} X_j = \Gamma_{ij}^k X_k. \tag{2.3}$$

We derive now an explicit formula for the Christoffel symbols of the Levi-Civita connection in terms of the fundamental matrix of the metric.

PROPOSITION 2.2. *Let (x,U) be a coordinate chart of M and $(X_1, \ldots, X_m)$ be the associated frame. Then*

$$\Gamma_{ij}^k = \frac{1}{2} g^{hk}(g_{jh,i} + g_{ih,j} - g_{ij,h}).$$

Proof. The Lie brackets $[X_i, X_j] = 0$, and hence, by the Koszul formula,

$$2\langle D_{X_i} X_j, X_h\rangle = g_{jh,i} + g_{ih,j} - g_{ij,h},$$

where (g_{ij}) is the fundamental matrix of g with respect to the coordinate chart x. By definition, the Christoffel symbols of D satisfy $D_{X_i} = \Gamma_{ij}^l X_l$. Therefore,

$$2\,\Gamma_{ij}^l g_{lh} = g_{jh,i} + g_{ih,j} - g_{ij,h}.$$

We have

$$\Gamma_{ij}^l g_{lh} g^{hk} = \Gamma_{ij}^l \delta_l^k = \Gamma_{ij}^k.$$

The asserted formula follows. □

EXAMPLES 2.3. 1) Let $(V, \langle\cdot,\cdot\rangle)$ be as in Example 1.1.1. With respect to the identification of vector fields with smooth maps $V \to V$, the standard derivative d is a symmetric connection on V. We write $d_X Y = dY\cdot X$ for the derivative of Y in the direction of X. For the constant metric on V as in (1.5),

$$X\langle Y,Z\rangle = \langle d_X Y, Z\rangle + \langle Y, d_X Z\rangle$$

for all vector fields X, Y, Z on V. Hence d is the Levi-Civita connection.

Let $B = (b_1, ..., b_m)$ be a basis of V, and set $\beta_{ij} = \langle b_i, b_j \rangle$, $1 \leq i, j \leq m$. The basis gives rise to a global coordinate chart $x_B : V \to \mathbb{R}^m$ of V defined by

$$x(p) = u \quad \text{iff } p = u^i b_i.$$

With respect to the above identification of tangent spaces with V, we have

$$g_{ij} = \langle X_i, X_j \rangle = \langle b_i, b_j \rangle = \beta_{ij}.$$

In particular, the functions g_{ij} are constant. It follows that the Christoffel symbols vanish identically, $\Gamma_{ij}^k \equiv 0$. We conclude that, for the associated frame,

$$D_{X_i} X_j \equiv 0,$$

and (again) that the standard derivative d is the Levi-Civita connection.

2) Consider Riemann's metric $\langle \cdot, \cdot \rangle_p$ as in Example 1.6, $\langle \cdot, \cdot \rangle_p = f_\kappa \langle \cdot, \cdot \rangle$. Let X, Y, Z be vector fields on V_κ, as usual considered as smooth functions $V \to V$. By the Koszul formula we have

$$\begin{aligned} 2f_\kappa \langle D_X Y, Z \rangle &= X(f_\kappa \langle Y, Z \rangle) + Y(f_\kappa \langle X, Z \rangle) - Z(f_\kappa \langle X, Y \rangle) \\ &\qquad - f_\kappa \langle X, [Y, Z] \rangle - f_\kappa \langle Y, [X, Z] \rangle + f_\kappa \langle Z, [X, Y] \rangle \\ &= X(f_\kappa) \cdot \langle Y, Z \rangle + Y(f_\kappa) \cdot \langle X, Z \rangle - Z(f_\kappa) \cdot \langle X, Y \rangle \\ &\qquad + f_\kappa \cdot \big\{ X \langle Y, Z \rangle + Y \langle X, Z \rangle - Z \langle X, Y \rangle \\ &\qquad\quad - \langle X, [Y, Z] \rangle - \langle Y, [X, Z] \rangle + \langle Z, [X, Y] \rangle \big\}. \end{aligned}$$

Now we know from Example 1) above that the term in braces on the right hand side is equal to $2 \langle d_X Y, Z \rangle$, where d denotes the standard derivative on V. On the other hand,

$$df_\kappa(p) = \frac{-16\kappa \langle p, \cdot \rangle}{(1 + \kappa \langle p, p \rangle)^3} = -\frac{4\kappa \langle p, \cdot \rangle}{1 + \kappa \langle p, p \rangle} \cdot f_\kappa(p),$$

and hence

$$D_X Y(p) = d_X Y(p) - \frac{2\kappa}{1 + \kappa \langle p, p \rangle} \big(\langle p, X \rangle Y + \langle p, Y \rangle X - \langle X, Y \rangle p \big). \tag{2.4}$$

3) In Example 1.3.2 we computed that the stereographic projection $h : Q_\alpha \cap U_0 \to W_{1/\alpha}$ as in (1.9) is an isometry. Hence the Levi-Civita conection on $W_{1/\alpha}$ corresponds via h to the Levi-Civita connection on $Q_\alpha \cap U_0$, see also Proposition 2.4 below. This description of the Levi-Civita connection of Q_α is not very useful. In [IS] we determine the Levi-Civita connection in a different way.

Proposition 2.4. *Let (M, g) and (N, h) be SR-manifolds and $f : M \to N$ be a local isometry. If $X_1, X_2 \in \mathcal{V}(M)$ and $Y_1, Y_2 \in \mathcal{V}(N)$ are f-related, then $D_{X_1} X_2$ and $D_{Y_1} Y_2$ are also f-related.*

Proof. Let $p \in M$ and $x = f \circ y$ and y be coordinate charts about p and $f(p)$ as in Proposition 1.2.2. Let η^1 and η^2 be the principal parts of Y_1 and Y_2 with respect to y. Then since Y_1 and Y_2 are f related to X_1 and X_2, respectively, $f \circ \eta^1$ and $f \circ \eta^2$ are the principal parts of X_1 and X_2 with respect to x, respectively.

Since $g_{ij} = h_{ij} \circ f$, it follows that the Christoffel symbols of g with respect to x are $\Gamma_{ij}^k \circ f$, where Γ_{ij}^k are the Christoffel symbols of h with respect to y. The claim follows. $\square$

3. Curvature

Let M be an SR-manifold nad D be the Levi-Civita connection of M. The *curvature tensor of* M is the curvature tensor associated to D, compare [CG]. This is a reasonable terminology, since D is determined by M.

PROPOSITION 3.1. *For all points $p \in M$ and tangent vectors $u, v, w, x, y \in T_pM$ we have*

1) $R(u,v)w = -R(v,u)w$

2) $R(u,v)w + R(v,w)u + R(w,u)v = 0$

3) $\langle R(x,y)u, v\rangle = -\langle R(x,y)v, u\rangle$

4) $\langle R(x,y)u, v\rangle = \langle R(u,v)x, y\rangle$

REMARK 3.2. Property 1) holds for all connections, 2) for all symmetric connections, 3) for all metric connections. Property 4) follows from the other properties.

Proof of Proposition 3.1. Property 1) is clear, the proof of 2) and 3) a straightforward computation. As for 4), we compute

$$\langle R(x,y)u,v\rangle = -\langle R(y,x)u,v\rangle = \langle R(x,u)y,v\rangle + \langle R(u,y)x,v\rangle$$

and similarly

$$\langle R(x,y)u,v\rangle = -\langle R(x,y)v,u\rangle = \langle R(y,v)x,u\rangle + \langle R(v,x)y,u\rangle.$$

Hence

$$\begin{aligned} 2\,\langle R(x,y)u,v\rangle = &\ \langle R(x,u)y,v\rangle + \langle R(u,y)x,v\rangle \\ &+\langle R(y,v)x,u\rangle + \langle R(v,x)y,u\rangle. \end{aligned}$$

Interchanging the roles of x and u, respectively y and v, we obtain

$$\begin{aligned} 2\,\langle R(u,v)x,y\rangle = &\ \langle R(u,x)v,y\rangle + \langle R(x,v)u,y\rangle \\ &+\langle R(v,y)u,x\rangle + \langle R(y,u)v,x\rangle. \end{aligned}$$

By 1) and 3), the right hand sides of the last two equations are equal, hence 4) follows. $\square$

EXAMPLES 3.3. 1) For $(V, \langle\cdot,\cdot\rangle)$ as in Example 1.1.1, consider the constant metric as in (1.5). In Example 2.3.1 we showed that the standard derivative d on V is the Levi-Civita connection of this metric. Now for all smooth vector fields on V,

$$d_X d_Y Z - d_Y d_X Z = d_{[X,Y]} Z,$$

hence $R = 0$. We also showed that the Christoffel symbols of the coordinate chart associated to a basis of V are constant. This is another way of seeing that $R = 0$.

2) For V_κ as in Example 1.1.2, consider Riemann's metric as in (1.6). Let $p \in V_\kappa$ and $u, v, w \in T_pV_\kappa$. To compute $R(u,v)w$, we let X, Y and Z be the constant vector fields on V with $X(p) = u$, $Y(p) = v$ and $Z(p) = w$. Then dX, dY, dZ and $[X,Y]$ vanish identically and $R_p(u,v)w = (R(X,Y)Z)(p)$. The straightforward calculation using the definition is rather tedious — it is usually not a good idea to work with the definition, but in this case it is still o.k. and a good exercise — the result is

$$R(u,v)w = \kappa \cdot \big(\langle v,w\rangle u - \langle u,w\rangle v\big)$$

We will see in the next section that this particular curvature tensor plays a special role.

3) Consider the quadric Q_α as in Example 1.1.3. Since the stereographic projection h as in (1.9) is an isometry for $\kappa = 1/\alpha$, we conclude that

$$R(u,v)w = \frac{1}{\alpha} \cdot \big(\langle v,w\rangle u - \langle u,w\rangle v\big),$$

see Proposition 3.4 below. There is another, more elegant way of determinig R, see [IS].

The following proposition is an immediate consequence of Proposition 2.4

PROPOSITION 3.4. *Let (M,g) and (N,h) be SR-manifolds and $f : M \to N$ be a local isometry. Let $p \in M$ and $u, v, w \in T_pM$. Then*

$$R_{f(p)}(f_*u, f_*v)f_*w = f_*(R_p(u,v)w). \quad \square$$

Let (x,U) be a coordinate chart of M. Let $(X_1, \ldots, X_m)$ be the associated frame of M over U, and let $g_{ij} = \langle X_i, X_j\rangle$ be the coefficients of the fundamental matrix of the SR-metric g of M. Define smooth functions $R^l_{ijk} : U \to \mathbb{R}$ by

$$R(X_i,X_j)X_k = R^l_{ijk}X_l. \tag{3.1}$$

By the definition of Christoffel symbols,

$$R^l_{ijk} = \Gamma^l_{jk,i} - \Gamma^l_{ik,j} + \big(\Gamma^l_{ih}\Gamma^h_{jk} - \Gamma^l_{jh}\Gamma^h_{ik}\big). \tag{3.2}$$

We also set

$$R_{ijkl} := \langle R(X_i,X_j)X_k, X_l\rangle = R^h_{ijk}\, g_{hl}. \tag{3.3}$$

3.1. **Sectional curvature.** We now derive a complete invariant for curvature tensors. Our computations are purely algebraic, we only use that $R = R_p$ is trilinear and satisfies Properties (1)–(4) above. Using (1) and (2) we have

$$\begin{aligned} 3R(u,v)w = {} & R(u,v+w)(v+w) - R(u,v)v - R(u,w)w \\ & -R(v,u+w)(u+w) + R(v,u)u + R(v,w)w. \end{aligned}$$

For $k(u,v) := \langle R(u,v)v,u\rangle$ we also have

$$2\,\langle R(u,v)v,w\rangle = k(u+w,v) - k(u,v) - k(w,v)\,.$$

Now the last two displayed equations yield an explicit formula

$$6\,\langle R(u,v)x,y\rangle = \dots$$

where the right hand side consists of eighteen terms of the form $k(w,z)$, where w and z are appropriate linear combinations of u, v, x and y. Hence R is determined by k.

Let a,b,c,d be real numbers and set $\tilde v = av + bw$ and $\tilde w = cv + dw$. Then

$$\begin{aligned} k(\tilde v,\tilde w) &= \langle R(\tilde v,\tilde w)\tilde w,\tilde v\rangle = (ad-bc)\langle R(v,w)\tilde w,\tilde v\rangle \\ &= (ad-bc)^2\langle R(v,w)w,v\rangle = (ad-bc)^2 k(v,w). \end{aligned}$$

For $\kappa \in \mathbb{R}$ and $u,v,w \in T_pM$ we set

$$R_\kappa(u,v)w = \kappa\big(\langle v,w\rangle u - \langle u,w\rangle v\big) \tag{3.4}$$

and

$$k_\kappa(u,v) := \langle R_\kappa(u,v)v,u\rangle = \kappa\big(\langle u,u\rangle\langle v,v\rangle - \langle u,v\rangle\langle v,u\rangle\big).$$

Note that R_κ is trilinear and satisfies the identities (1)–(4) in Proposition 3.1 . Hence

$$k_\kappa(\tilde v,\tilde w) = (ad-bc)^2 k_\kappa(v,w)$$

for $\tilde v,\tilde w \in T_pM$ as above. We conclude that k and k_κ transform in the same way when (v,w) is replaced by $(\tilde v,\tilde w)$. To get an invariant, we take the quotient of k and k_1. To avoid zero in the denominator, we take the following precaution.

Let $\sigma \subset T_pM$ be a *tangential 2-plane*, that is, a 2-dimensional linear subspace of T_pM. We say that σ is *non-degenerate* if the restriction of $\langle\cdot,\cdot\rangle$ to σ is non-degenerate. If v,w is a basis of σ, then σ is a non-degenerate subspace of T_pM if and only if $k_1(v,w) \neq 0$. If the metric of M is Riemannian, then $\langle\cdot,\cdot\rangle$ is positive definite and any tangential 2-plane in T_pM is non-degenerate.

Now let $\sigma \subset T_pM$ be a non-degenerate tangential 2-plane in T_pM. By what we said above, the quotient

$$K(\sigma) = \frac{k(v,w)}{k_1(v,w)} = \frac{\langle R(v,w)w,v\rangle}{\langle v,v\rangle\langle w,w\rangle - \langle v,w\rangle^2} \tag{3.5}$$

is independent of the basis v,w of σ. The number $K(\sigma)$ is called the *sectional curvature* of σ. The sectional curvature is defined for all tangential 2-planes if M is Riemannian and for all non-degenerate tangential 2-planes otherwise. In the Examples 2.14 and 2.15 above we showed that the sectional curvature is identically equal to κ.

Now $k : T_pM \times T_pM \to \mathbb{R}$ is a polynomial, hence determined by its values on the set of pairs $v,w \in T_pM$ with $k_1(v,w) \neq 0$. Therefore we have the following result.

PROPOSITION 3.5. *The sectional curvature K of (non-degenerate) tangential 2-planes in T_pM, $p \in M$, determines the curvature tensor R_p. In particular, if $K(\sigma) \equiv \kappa$, then $R_p = R_\kappa$.*

3.2. **Ricci and scalar curvature.** The *Ricci tensor* of an SR-manifold is a weaker invariant of the SR-metric than the curvature tensor. It is defined by a trace,

$$\mathrm{ric}_p(v,w) = \mathrm{tr}\,\big(u \mapsto R_p(u,v)w\big). \tag{3.6}$$

By Proposition 3.1 above ric_p is symmetric. If $b_1, ..., b_m$ is an ON-base of T_pM, then

$$\mathrm{ric}_p(v,w) = \sum_{i=1}^{m} \varepsilon_i \, \langle R(b_i,v)w, b_i\rangle. \tag{3.7}$$

For $\langle v,v\rangle \neq 0$ the quotient

$$\frac{\mathrm{ric}_p(v,v)}{\langle v,v\rangle} \tag{3.8}$$

is called the *Ricci curvature* in the direction of v. Now there is an endomorphism Ric_p of T_pM such that

$$\mathrm{ric}_p(v,w) = \langle \mathrm{Ric}_p\, v, w\rangle \tag{3.9}$$

for all $v, w \in T_pM$. The trace of the Ricci tensor is called the *scalar curvature*, $\mathrm{scal}(p) := \mathrm{tr}\,\mathrm{Ric}_p$. With respect to an ON-base of T_pM, the scalar curvature is

$$\mathrm{scal}(p) = \sum_{i,j=1}^{m} \varepsilon_i\varepsilon_j \, \langle R(b_j,b_i)b_i, b_j\rangle. \tag{3.10}$$

With respect to a local coordinate system, Ricci tensor and scalar curvature are given by

$$\mathrm{ric}_{ij} := \mathrm{ric}(X_i, X_j) = R^l_{lij} \quad \text{and} \quad \mathrm{scal} = g^{ij}\,\mathrm{ric}_{ij}\,. \tag{3.11}$$

3.3. **Curvature in dimension 3.** Let V be an m-dimensional real vector space and $g = g_V$ be a non-degenerate symmetric bilinear form on V. Let $\mathcal{C}$ be the vector space of all quadrilinear maps $R : V^4 \to \mathbb{R}$ which satisfy Properties (1)–(4) in Proposition 3.1, where we write $\langle R(u,v)x,y\rangle = R(u,v,x,y)$. We call elements of $\mathcal{C}$ *algebraic curvature tensors* on V. Our arguments above show that any $R \in \mathcal{C}$ is determined by its sectional curvature $K = K_R$.

Let h and k be symmetric bilinear forms on V. Set

$$\begin{aligned}(h \odot k)(u,v,x,y) = {} & h(v,x)k(u,y) - h(u,x)k(v,y)\\ & + k(v,x)h(u,y) - k(u,x)h(v,y).\end{aligned}$$

Then $h \odot k \in \mathcal{C}$. We have $g \odot g = R_2 = 2 \cdot R_1$.

We now consider the case $m = 3$. Using Properties (1), (3) and (4) of curvature tensors, we get $\dim \mathcal{C} \leq 6$. With a given number s and a given symmetric bilinear form b on T_pM with $\mathrm{tr}\, b = 0$ we associate

$$R = \frac{s}{12}\, g \odot g + b \odot g \in \mathcal{C}.$$

Then the Ricci tensor of R is $b + \frac{s}{3}g$, and the scalar curvature is s. Therefore, the map $(s, b) \mapsto R$ is injective, hence $\dim \mathcal{C} = 6$, and the map is an isomorphism. Thus

$$R = \frac{s}{12}\, g \odot g + (\mathrm{ric} - \frac{s}{3}g) \odot g.$$

Hence for a 3-dimensional SR-manifold, the curvature tensor is determined by the Ricci tensor. For $\dim M \geq 4$ this does not hold anymore. The remainder W in the decomposition

$$R = \frac{s}{2n(n-1)}\, g \odot g + \frac{1}{n-2}(\mathrm{ric} - \frac{s}{n}g) \odot g + W,$$

is called the *Weyl-Tensor*. The Weyl tensor is related to *conformal geometry*.

Appendix A. Non-degenerate bilinear forms

Let V be a real vector space of dimension m with a non-degenerate symmetric bilinear form g. The latter is usually also denoted $\langle\cdot,\cdot\rangle$. If $B = \{b_1, \ldots, b_m\}$ is a basis of V and

$$\beta_{ij} := \langle b_i, b_j\rangle, \quad 1 \le i, j \le m, \tag{A.1}$$

denotes the *fundamental matrix* of g with respect to B, then

$$\langle v, w\rangle = \beta_{ij} v^i w^j \tag{A.2}$$

for $v = v^i b_i, w = w^i b_i \in V$. We say that a basis B as above is an *orthonormal basis (ON-basis)* if

$$\langle b_i, b_j\rangle = \varepsilon_i \delta_{ij} \tag{A.3}$$

with $\varepsilon_i \in \{+1, -1\}$ for all i.

The *index* of $\langle\cdot,\cdot\rangle$ is the maximal dimension of a linear subspace of V on which $\langle\cdot,\cdot\rangle$ is negative definite. In terms of an ON-basis as above, the index is equal to the number of indices i with $\varepsilon_i = -1$. In particular, $\langle\cdot,\cdot\rangle$ is positive definite if and only if it has index 0.

Examples A.1. 1) The model example is the space $\mathbb{R}^m_k$. Here k is an integer in $\{0, ..., m\}$ and $\mathbb{R}^m_k$ denotes $\mathbb{R}^m$ equipped with the bilinear form

$$\langle v, w\rangle = -\sum_{i=1}^{k} v^i w^i + \sum_{i=k+1}^{m} v^i w^i\,. \tag{A.4}$$

For $k = 0$, this is m-dimensional *Euclidean space* and for $k = 1$, m-dimensional *Minkowski space.*

For any non-degenerate symmetric bilinear form g on a vector space of dimension m, there is an orthonormal basis $B = (b_1, \ldots, b_m)$ of V with $\langle b_i, b_j\rangle = -\delta_{ij}$ for $i \le k$ and $\langle b_i, b_j\rangle = -\delta_{ij}$ for $i > k$, where k is the index of g. Then the linear map $V \to \mathbb{R}^m_k$, which maps b_i to the i-th unit vector in $\mathbb{R}^m_k$, is an isomorphism between (V, g) and $\mathbb{R}^m_k$.

2) Consider the vector space $V = \mathbb{R}^{n\times n}$ of real $(n\times n)$-matrices. The dimension of V is $m = n^2$. Recall that the trace satisfies

$$\operatorname{tr} A = \sum_{i=1}^{m} a^i_i = \operatorname{tr} A^* \quad \text{and} \quad \operatorname{tr}(AB) = \sum_{i,j=1}^{m} a^j_i b^i_j = \operatorname{tr}(BA), \tag{A.5}$$

where $A = (a^j_i), B = (b^j_i) \in \mathbb{R}^{n\times n}$, and A^* denotes the transposed matrix of A. In particular,

$$\langle A, B\rangle = \operatorname{tr}(A^* B) \tag{A.6}$$

is a symmetric bilinear form on $R^{n\times n}$. Now (A.6) implies that $\langle A, B\rangle$ if A is symmetric, $A = A^*$, and B is skew symmetric, $B = -B^*$. Clearly, $\langle\cdot,\cdot\rangle$ is negative definite on the subspace of skew symmetric matrices and positive definite on the subspace of symmetric matrices. Since any $A \in \mathbb{R}^{n\times n}$ can be written as a sum of a symmetric and a skew symmetric matrix,

$$A = \frac{A + A^*}{2} + \frac{A - A^*}{2},$$

$R^{n\times n}$ is the orthogonal sum of these subspaces. Therefore, $\langle\cdot,\cdot\rangle$ is a non-degenerate symmetric bilinear form of index $\binom{n}{2}$ on $\mathbb{R}^{n\times n}$.

Appendix B. Two technical lemmas

There are some lemmas of a more technical nature which we use over and over again. We also need modifications of these lemmas, the arguments underlying the proofs are useful tools.

Lemma B.1. *1) Given $p \in M$ and a neighborhood U of p in M, there exists a function $\varphi \in \mathcal{F}(M)$ such that $0 \le \varphi \le 1$, $\operatorname{supp}\varphi \subset U$ and such that $\varphi(q) = 1$ for all q in a (small) neighborhood of p.*
2) Given $p \in M$ and a tangent vector $v \in T_pM$, there is a smooth vector field X on M with $X(p) = v$.
3) Let $W \subset M$ be open, $X \in \mathcal{V}(W)$, and $p \in W$. Then there is $Y \in \mathcal{V}(M)$ with $X(q) = Y(q)$ for all q in a neighborhood of p.

Proof. 1) By replacing U by a smaller neighborhood of p if necessary, we can and will assume that there is a coordinate chart $x : U \to U'$. We arrange x such that $x(p) = 0$ and let $\psi : \mathbb{R}^m \to \mathbb{R}$ be a smooth function with $0 \le \psi \le 1$, $\operatorname{supp}\psi \subset U'$ and such that $\psi(u) = 1$ for all u in a (small) neighborhood of 0. Then $\varphi : M \to \mathbb{R}$, $\varphi = \psi \circ \psi$ on U and $\psi = 0$ otherwise satisfies the assertions. Then $\varphi = 0$ in a neighborhood of the boundary of U, hence φ is smooth.

2) Choose a coordinate chart $x : U \to U'$ about p. Let $X_1 \dots, X_m$ be the associated frame and $\xi \in \mathbb{R}^m$ be the principal part of v with respect to x, $v = \xi^i X_i(p)$. Choose φ as in 1) and define

$$Y(q) = \begin{cases} \varphi\xi^i X_i(q) & \text{for } q \in U, \\ 0 & \text{otherwise.} \end{cases}$$

Then $Y = 0$ in a neighborhood of the boundary of U, hence Y is smooth.

3) We use the same argument as in 2): Choose the coordinate chart x such that $U \subset W$. Now the principal part of X with respect to x is a map $\xi : U \to \mathbb{R}^m$, all we do now is to replace, in the definition of Y, the constants ξ^i by the values $\xi(q)$. □

Lemma B.2. *Let $\Phi : \mathcal{V}(M) \to \mathcal{F}(M)$ be a linear map which is* tensorial*, i.e.,*

$$\Phi(\varphi X) = \varphi\Phi(X) \quad \textit{for all } \varphi \in \mathcal{F}(M) \textit{ and } X \in \mathcal{V}(M).$$

Then there is a smooth 1-form ω with

$$\Phi(X)(p) = \omega_p(X(p)) \quad \textit{for all } X \in \mathcal{V}(M) \textit{ and } p \in M.$$

Proof. Let $p \in M$ and X and Y be smooth vector fields on M such that $X(q) = Y(q)$ for all q in a neighborhood V of p. It suffices to show that $\Phi(X)(p) = \Phi(Y)(p)$.

In a neighborhood $U \subset V$ of p, choose a local frame $(X_1, \dots, X_m)$ of TM and a function $\varphi \in \mathcal{F}(M)$ with $\varphi(p) = 1$ and $\operatorname{supp}(\varphi) \subset U$. Then $\varphi \cdot X_i$ is a smooth vector field when extended by zero outside U, $1 \le i \le m$.

By assumption, the principal parts of X and Y coincide, $X = \xi^i X_i = Y$ on U. The functions $\varphi\xi^i$ are smooth on M when extended by zero outside U and

$$\varphi^2 \cdot X = (\varphi\xi^i) \cdot (\varphi X_i) = \varphi^2 \cdot Y.$$

Hence

$$\Phi(X)(p) = \varphi^2(p) \cdot \Phi(X)(p) = \Phi(\varphi^2 \cdot X)(p) = \Phi(Y)(p).$$

□

Acknowledgments

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