ESSENTIAL SPECTRUM AND SMALL EIGENVALUES
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Report on joint work with Panagiotis Polymerakis.

1. Setup

M complete Riemannian mfld, E Riemannian vb over M with metric conn ∇, $A = \nabla^*\nabla + V$ Schrödinger operator with symmetric potential; throughout $V \geq b$. Examples: Laplacian Δ on functions, Bochner-Laplacian $\nabla^*\nabla$, Hodge-Laplacian $\Delta_{\mathrm{H}} = (d + d^*)^2$ on differential forms if curvature K_M of M is bounded.

We say that A is *homogeneous* if the group of automorphisms of E, which preserve A and act isometrically on the base M, is transitive on M. Then M is homogeneous. Conversely, if M is homogeneous, then Δ and Δ_H are homogeneous.

1.1. **Spectrum.** A is *essentially self-adjoint*, see e.g., [1, Thm A.17]; i.e., its graph closure in $L^2(E)$, also denoted by A, is self-adjoint. *Spectrum* $\sigma(A)$ of A: $\lambda \in \sigma(A)$ iff there is a *Weyl sequence* for λ: a sequence of $u_n \in C_c^\infty(E)$ such that $\|u_n\|_2 \to 1$ and $\|(A - \lambda)u_n)\|_2 \to 0$; we have

$$[b, \infty) \supseteq \sigma(A) = \underset{\text{discrete}}{\sigma_d(A)} \,\dot\cup\, \underset{\text{essential}}{\sigma_{\mathrm{ess}}(A)},$$

where $\lambda \in \sigma_{\mathrm{ess}}(A)$ if $A - \lambda$ is not a Fredholm operator. Note:
1) $\lambda \in \sigma_d(A)$, then λ eigenvalue of A of finite multiplicity; λ is isolated in $\sigma(A)$.
2) $\lambda \in \sigma_{\mathrm{ess}}(A)$ iff there is a Weyl sequence (u_n) for λ such that $\operatorname{supp} u_n$ escapes to infinity. In particular, $\sigma(A) = \sigma_{\mathrm{ess}}(A)$ if M is non-compact and A is homogeneous.

Donnelly '80 determined $\sigma(\Delta_{\mathrm{H}})$ for $H^m_{\mathbb{R}}$, Pedon '98/99/05 for $H^k_{\mathbb{F}}$ with $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$; only partially known for $H^2_{\mathbb{O}}$. (Carron: Latter may be known to Olbrich). Bunke '91 determined $\sigma(\Delta_{\mathrm{spin}})$ for $H^m_{\mathbb{R}}$; Bär '92 determined bottom of $\sigma(\Delta_{\mathrm{spin}})$ for $H^m_{\mathbb{R}}$, method more elementary than Bunke's.

1.2. **Geometry.** Throughout $-1 \leq K_M \leq -a^2$ with $0 < a \leq 1$, $M = \Gamma\backslash X$;
X_∞ the *sphere at infinity*, $\bar X = X \cup X_\infty$,
$\Lambda \subseteq X_\infty$ the limit set of Γ, $C_\Lambda \subseteq X$ the closed convex hull of Λ,
$C = C_M = \Gamma\backslash C \subseteq M$ the *convex core* of M. M *convex cocompact: C is compact.*
C is a deformation retract of M along shortest geodesic connections.

Remarks 1.1. 1) M compact or $|M| < \infty$, then $\Lambda = X_\infty$ and therefore $C = M$.
2) If M is a surface of finite type, then C is a convex subset of M with closed geodesics as boundary components, cutting away *expanding tubes*. Ends of C are cuspidal.

Definition 1.2 (Bowditch '95)**.** Say that M is *geometrically finite* if $|U_r(C)| < \infty$ for the tube of some–or any–radius $r > 0$ about C.

In his article [4], Bowditch formulates five definitions. Four of them are equivalent, but the classical one from real hyperbolic geometry concerning fundamental domains does not extend to the variable curvature setting.

2. Essential spectrum

Bottom of spectrum and essential spectrum: $\lambda_0(A) \le \lambda_{\text{ess}}(A)$.
There is only discrete spectrum below $\lambda_{\text{ess}}(A)$, if any.
If M is compact, then $\lambda_{\text{ess}}(A) = \infty$, whereas always $\lambda_0(A) < \infty$.
if M is non-compact and A is homogeneous, then $\lambda_0(A) = \lambda_{\text{ess}}(A)$.

Mazzeo-Phillips '90 determine $\sigma_{\text{ess}}(\Delta_H)$ for geometrically finite $\Gamma\backslash H^m_{\mathbb{R}}$;
Carron-Pedon '04 determine $\sigma(\Delta_H)$ resp. lower bounds for it, for $\Gamma\backslash H^m_{\mathbb{F}}$,
assuming upper bounds on critical exponent of Γ,
related recent work by Samuel Tapie and coauthors;
Bunke-Olbrich '00 discuss convex cocompact case.

Proposition 2.1. *If $\Lambda \ne X_\infty$ and A_X (the lift of A to X) is homogeneous, then*

$$\sigma(A_X) = \sigma_{\text{ess}}(A_X) \subseteq \sigma_{\text{ess}}(A) \quad \textit{and} \quad \lambda_0(A_X) = \lambda_{\text{ess}}(A_X) \ge \lambda_{\text{ess}}(A).$$

Proof. (Following Carron-Pedon [6].) Recall that Γ acts properly discontinuously on $\Omega = X_\infty \setminus \Lambda$, and let $x \in \Omega$ have trivial isotropy group. Choose a sequence $U_1 \supset U_2 \supset \dots$ of neighborhoods of x in $\bar{X}$ such that $\cap U_n = \{x\}$ and such that $\gamma U_1 \cap U_1 \ne \emptyset$ implies that $\gamma = \text{id}$.

Use that $\lambda \in \sigma(A_X)$ if and only if there is a Weyl sequence $u_n \in C^\infty_c(E_X)$ for λ. Now for each n, translate u_n to have support in U_n. Then u_n can be pushed down to E over M and yields a Weyl sequence for λ such that the supports escape to infinity. □

Theorem 2.2 (BP '21)**.** *If M is geometrically finite, then*

$$\lambda_{\text{ess}}(A_X) \le \lambda_{\text{ess}}(A).$$

Moreover, if $\|\nabla R_M\|_\infty < \infty$, then

$$\sigma_{\text{ess}}(A_X) \supseteq \sigma_{\text{ess}}(A).$$

In both cases, equality holds if $|M| = \infty$ and A_X is homogeneous.

Sketch of proof. Draw figure showing the convex core C, cover thick part of C by finitely many small balls B_i to obtain a locally finite cover U_i of C, where the B_i are complemented by the finitely many (neighborhoods of) the parabolic ends of C. Choose smooth functions ψ_i such that the ψ_i^2 are an associated partition of unity. (A trick we learned from Jialun Li [10].) Let $\varphi_i = \psi_i \circ \pi$, where $\pi\colon M \to C$ is the metric projection. Then

$$\langle Au, u\rangle_2 = \sum_i \langle A(\varphi_i u), \varphi_i u\rangle - \sum_i \int |\nabla\varphi_i|^2 |u|^2.$$

Therefore

$$\begin{aligned}
\frac{\sum\langle A(\varphi_i u), \varphi_i u\rangle_2}{\sum \|\varphi_i u\|_2^2} &= \frac{\langle Au, u\rangle_2 + \sum_i \int |\nabla\varphi_i|^2 |u|^2}{\sum_i \int \varphi_i^2 |u|^2} \\
&= \frac{\langle Au, u\rangle_2 + \sum_i \int |\nabla\varphi_i|^2 |u|^2}{\|u\|_2^2} \\
&\le \frac{\langle Au, u\rangle_2}{\|u\|_2^2} + \sum \|\nabla\varphi_i\|^2_{\operatorname{supp} u, \infty}.
\end{aligned}$$

In particular, if $u \ne 0$, then there is an i with $\varphi_i u \ne 0$ and

$$\operatorname{Ray}_A(\varphi_i u) \le \operatorname{Ray}_A(u) + \sum \|\nabla\varphi_i\|^2_{\operatorname{supp} u, \infty}.$$

The first assertion then follows, for the B_i by transplantation, for the parabolic ends by Polymerakis [16] on amenable coverings. The second assertion is more intricate, as is clear from the formulation. □

3. Small eigenvalues

In our cases below always: $\lambda_0(A_X) \leq \lambda_{\mathrm{ess}}(A)$.
Say (in this talk) that an eigenvalue λ of A is *small* if $\lambda \leq \lambda_0(A_X)$.

Example 3.1. If $X = H^2_{\mathbb{R}}$, then an eigenvalue $\lambda \leq 1/4$ is small.

Let $N_A(\lambda)$ be the number of small eigenvalues $\leq \lambda \leq \lambda_0(A_H)$.

Remark 3.2. If $\dim M = m$ and $-1 \leq K_M \leq -a^2$, where $0 < a \leq 1$, then, for Δ the Laplacian on functions,

$$a^2(m-1)^2/4 \underset{\text{McKean}}{\leq} \lambda_0(\Delta_X) \underset{\text{Cheng}}{\leq} (m-1)^2/4$$

More restrictive notion of smallness in this case would be $\lambda \leq a^2(m-1)^2/4$.

Theorem 3.3 (Buser-Colbois-Dodziuk '93 for the Laplacian on functions)**.**
If $|M| < \infty$ and $-1 \leq K_M \leq -a^2$, then $N_\Delta(a^2(m-1)^2/4) \leq C(m,a)|M|$.

Extending results of Otal-Rosas '09 and earlier ones of Buser, Schmutz:

Theorem 3.4 (B-Matthiesen-Mondal '17 for the Laplacian on functions)**.**
If M is a surface of finite type, then $N_\Delta(\lambda_0(\Delta_X)) \leq -\chi(M)$.

Remark 3.5. In Theorem 3.4, if $|M| < \infty$ and $K_M \geq -1$, then $-2\pi\chi(M) = \int K_M \leq |M|$. Hence linearity in $|M|$ is/seems optimal in Theorem 3.3 with $C(2,a) = 1/2\pi$. ($a^2(m-1)^2/4$ versus $\lambda_0(\Delta_X)$)

Theorem 3.6 (Hamenstädt '04 for the Laplacian on functions)**.**
If M is geometrically finite with $|M| = \infty$, then
1) $\lambda_{\mathrm{ess}}(\Delta) \geq a^2(m-1)^2/4$.
2) $N_\Delta(a^2(m-1)^2/4 - \varepsilon) \leq e^{C(m,a,\varepsilon)|U_1(C)|}$.

Remark 3.7. For $X = H^k_{\mathbb{F}}$, with $\min K_H = -1$, she can substitute $a^2(m-1)^2/4$ by $\lambda_0(\Delta_X)$ in 2), which is better in the case $\mathbb{F} \neq \mathbb{R}$.
Recall that $\lambda_0(\Delta_X) = h^2/4$ for $X = H^k_{\mathbb{F}}$, where h is the volume entropy.

Theorem 3.8 (J. Li '20 for the Laplacian on functions)**.**
If $M = \Gamma\backslash H^k_F$ and M is geometrically finite, then $N_\Delta(\lambda_0(\Delta_X)-) < \infty$.

Theorem 3.9 (BP '21 for the Laplacian on functions)**.**
If M is geometrically finite, then $N_\Delta(\lambda_0(\Delta_X) - \varepsilon) \leq C(m,a,\varepsilon)|U_1(C)|$.

Remark 3.10. In [3], we have the analogous result in the case of operators A as above, where we need that E_X is an associated bundle via an orthogonal representation θ of the orthogonal group, with induced metric and connection, to get

$$N_A(\lambda_0(A_X) - \varepsilon) \leq C(m, a, \|\theta_*\|, \varepsilon)\,\mathrm{rk}(E)|U_1(C)|.$$

4. Open problems and questions

(1) In Theorem 3.3, can one substitute $a^2(m-1)^2/4$ by $\lambda_0(\Delta_X)$?
For this it would be sufficient to get that, say, the first non-zero Neumann eigenvalue of domains, which they call pieces of cheese, is at least $\lambda_0(\Delta_X)$.

(2) In Remark 3.10, can one consider more general vector bundles instead of associated ones?
In our arguments, we use e.g. that the $(\mathrm{rk}(E)+1)^{st}$ Neumann eigenvalue of A restricted to small balls in M is at least $C(\dim M)\lambda_0(A_X)$.

References

[1] W. Ballmann and P. Polymerakis, Bottom of spectra and coverings. Surv. Differ. Geom. **23** (2020), 1–33.

[2] W. Ballmann and P. Polymerakis, On the essential spectrum of differential operators over geometrically finite orbifolds. J. Differential Geometry, accepted for publication.

[3] W. Ballmann and P. Polymerakis, Small eigenvalues of Schrödinger operators over geometrically finite manifolds. MPIM Preprint 2021-30, arxiv: 2106.13437

[4] B. H. Bowditch, Geometrical finiteness with variable negative curvature. *Duke Math. J.* **77** (1995), no. 1, 229–274.

[5] Peter Buser, Bruno Colbois, and Józef Dodziuk, *Tubes and eigenvalues for negatively curved manifolds*, J. Geom. Anal. **3** (1993), no. 1, 1–26.

[6] G. Carron and E. Pedon, *On the differential form spectrum of hyperbolic manifolds*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **3** (2004), no. 4, 705–747.

[7] S. Y. Cheng, Eigenvalue comparison theorems and its geometric applications. *Math. Z.* **143** (1975), no. 3, 289–297.

[8] H. Donnelly, *The differential form spectrum of hyperbolic space*, Manuscripta Math. **33** (1980/81), no. 3-4, 365–385.

[9] U. Hamenstädt, Small eigenvalues of geometrically finite manifolds. *J. Geom. Anal.* **14** (2004), no. 2, 281–290.

[10] J. Li, Finiteness of small eigenvalues of geometrically finite rank one locally symmetric manifolds. *Math. Res. Lett.* **27** (2020), no. 2, 465–500.

[11] H. P. McKean, *An upper bound to the spectrum of Δ on a manifold of negative curvature*, J. Differential Geometry **4** (1970), 359–366.

[12] J.-P. Otal and E. Rosas, Pour toute surface hyperbolique de genre g, $\lambda_{2g-2} > 1/4$. *Duke Math. J.* **150** (2009), no. 1, 101–115.

[13] E. Pedon, *Analyse harmonique des formes différentielles sur l'espace hyperbolique réel. I. Transformation de Poisson et fonctions sphériques*, C. R. Acad. Sci. Paris Sér. I Math. **326** (1998), no. 6, 671–676.

[14] ———, *Harmonic analysis for differential forms on complex hyperbolic spaces*, J. Geom. Phys. **32** (1999), no. 2, 102–130.

[15] ———, *The differential form spectrum of quaternionic hyperbolic spaces*, Bull. Sci. Math. **129** (2005), no. 3, 227–265.

[16] P. Polymerakis, *On the spectrum of differential operators under Riemannian coverings,* J. Geom. Anal. **30** (2020), no. 3, 3331–3370.