The range of a connection and a Calabi operator for locally symmetric spaces

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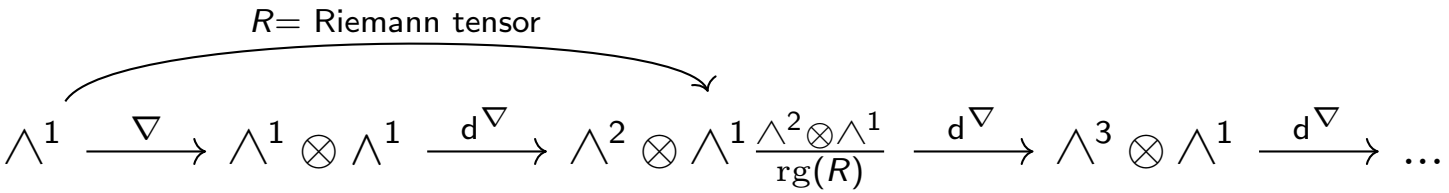
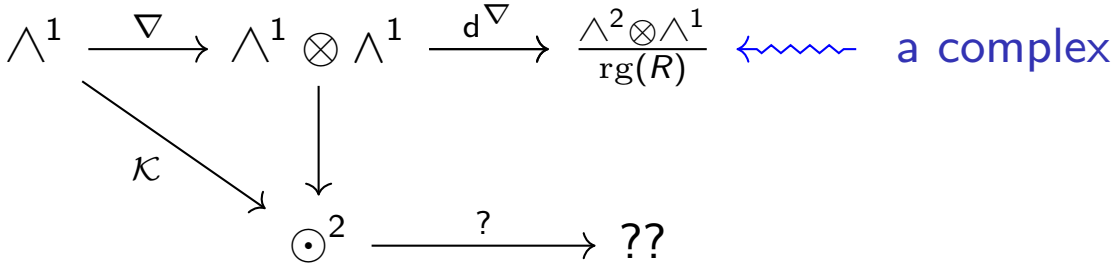
The Killing operator

Let (M, g) be a semi-Riemannian manifold.

- $\wedge^1 = T^*M$, $\wedge^k = k$-forms, $\odot^2 =$ symmetric BLFs on TM,
- $\boxplus = \{R_{abcd} \in \wedge^2 \otimes \wedge^2 \mid R_{[abc]d} = 0\}$ algebraic curvature tensors.





where $\mathcal{K} : \xi_a \longmapsto \nabla_{(a}\xi_{b)}$, or $\mathcal{K}(X^\flat) = \frac{1}{2}\mathcal{L}_X g$, is the Killing operator.

- Local integrability conditions for the range of $\mathcal{K}$?
- Range of $\mathcal{K}$ = infinitesimal isometric perturbations of g.
- Compatibility operator $\mathcal{C}$: for $h \in \odot^2$, $h \in \mathrm{rg}(\mathcal{K}) \iff h \in \ker(\mathcal{C})$

The Calabi operator

Theorem (Calabi '61)

On a space of constant sectional curvature, there is a 2^{nd} order differential operator $\mathcal{C} : \odot^2 \longrightarrow \boxplus$, such that

$$\wedge^1 \xrightarrow{\mathcal{K}} \odot^2 \xrightarrow{\mathcal{C}} \boxplus, \qquad \text{with } \mathcal{C}(h) = \mathrm{pr}_{\boxplus}(\nabla\nabla h) - \tfrac{1}{2} h \cdot R,$$

is locally exact.

- $\mathrm{pr}_{\boxplus} : \wedge^1 \otimes \wedge^1 \otimes \odot^2 \to \boxplus$ is the projection such that

$$h_{ab} \overset{\mathrm{pr}_{\boxplus} \circ \nabla^2}{\longmapsto} \nabla_{(a}\nabla_{c)} h_{bd} - \nabla_{(b}\nabla_{c)} h_{ad} - \nabla_{(a}\nabla_{d)} h_{bc} + \nabla_{(b}\nabla_{d)} h_{ac}.$$

- $h \cdot R$ is the action of the symmetric endomorphism h on R,

$$(h \cdot R)_{abcd} = 2\left(R_{ab}{}^{e}{}_{[c} h_{d]e} + R_{cd}{}^{e}{}_{[a} h_{b]e}\right).$$

$$h_{ab} \overset{\mathcal{C}}{\mapsto} \nabla_{(a}\nabla_{c)} h_{bd} - \nabla_{(b}\nabla_{c)} h_{ad} - \nabla_{(a}\nabla_{d)} h_{bc} + \nabla_{(b}\nabla_{d)} h_{ac} - R_{ab}{}^{e}{}_{[c} h_{d]e} - R_{cd}{}^{e}{}_{[a} h_{b]e}.$$

The Calabi operator on semi-Riemannian manifolds

- On an a general semi-Riemannian manifold we can still consider $\mathcal{C}(h) = \mathrm{pr}_{\boxplus}(\nabla\nabla h) - \frac{1}{2} h \cdot R$, but $\wedge^1 \xrightarrow{\mathcal{K}} \odot^2 \xrightarrow{\mathcal{C}} \boxplus$ is no longer a complex,

$$\mathcal{C} \circ \mathcal{K}(\xi) \;=\; \mathcal{R}(\mathrm{d}\xi) - \nabla_\xi R, \tag{1}$$

where $\mathcal{R} : \wedge^2 \to \boxplus$ is defined by the action of skew endomorphisms on R,

$$\mathcal{R}(\mu)_{abcd} = -(\mu \cdot R)_{abcd} = 2\left(R_{ab}{}^{e}{}_{[c}\mu_{d]e} + R_{cd}{}^{e}{}_{[a}\mu_{b]e}\right).$$

- $\mathcal{R} = 0 \iff R$ is of constant sectional curvature.
- Relation to the deformation operator:

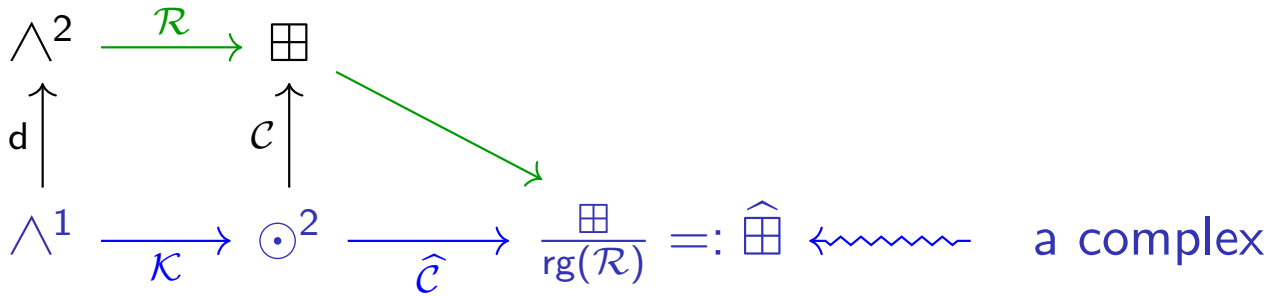
  If $g_t := \phi_t^* g$, then $R_t = \phi_t^* R$, then

$$\begin{aligned}
\frac{dR_t}{\mathrm{d}t} \;=\; \mathcal{L}_\xi R \;&=\; \nabla_\xi R - (\nabla\xi)\cdot R \;=\; \nabla_\xi R - (\mathcal{K}\xi)\cdot R - \mathcal{R}(\mathrm{d}\xi) \\
&=\; -\mathcal{C}\circ\mathcal{K}(\xi) + \mathcal{R}(\mathrm{d}\xi) \;\underline{-(\mathcal{K}\xi)\cdot R}\; - \mathcal{R}(\mathrm{d}\xi) \\
&=\; \underbrace{-\Big(\mathrm{pr}_{\boxplus}(\nabla\nabla h) \;\underline{+\; \tfrac{1}{2} h\cdot R}\Big)}_{\text{deformation operator } \mathcal{D}}, \qquad \text{where } h = \mathcal{K}(\xi).
\end{aligned}$$

  I.e. if $\widetilde{g} = g + \epsilon\, h$, then $\widetilde{R} = R + \epsilon\, \mathcal{D}(h) + \mathcal{O}(\epsilon^2)$.

The Calabi operator on locally symmetric spaces

On a locally symmetric space, $\nabla R = 0$, the diagram commutes



- $\widehat{\mathcal{C}}$ is of 2nd order, a *"Calabi operator"*.
- When is this locally exact?

Theorem (Costanza, Eastwood, L & McMillan '21,'22)

On a locally symmetric Riemannian or Lorentzian space (M, g),
$\bigwedge^1 \xrightarrow{\mathcal{K}} \odot^2 \xrightarrow{\widehat{\mathcal{C}}} \widehat{\boxplus}$ *is locally exact, unless (M, g) admits a parallel vector field and a Hermitian factor in its local de Rham decomposition.*

Previous results on compatibility operators for $\mathcal{K}$

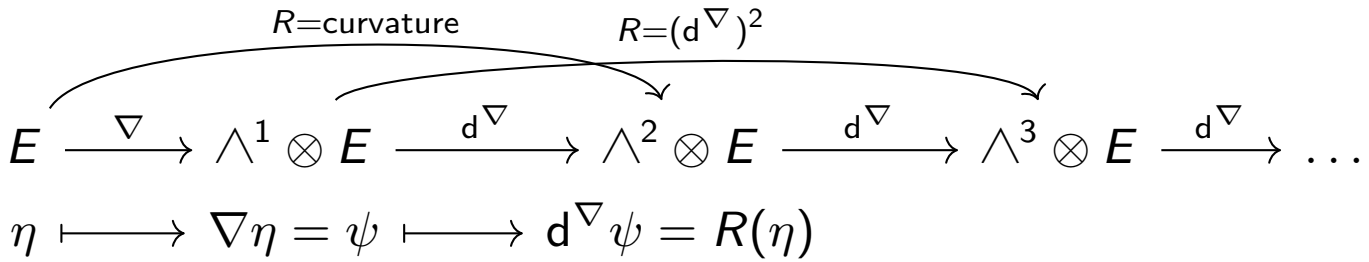
- Gasqui & Goldschmidt '83, 3^{rd} order compatibility operator on Riemannian symmetric spaces
- Khavkine '19, compatibility operator of 4^{th} order for Schwarzschild (warped product with constant curvature)
- Aksteiner, Andersson, Bäckdahl, Khavkine & Whiting '21, compatibility operator of higher order for Kerr

Our proof uses the prolongation of the Killing equation to a connection on $E = \wedge^1 \oplus \wedge^2$ whose parallel sections correspond to Killing vector fields (c.f. Christian's prolongation of the Killing spinor equation to parallel spinors on the cone).

$\rightsquigarrow$ Killing connection and its "exactness"

Exact connections

Let $(E \to M, \nabla)$ be a vector bundle with connection ∇. Twisted de Rham:



- We say the connection ∇ is *exact*, if for each $\psi \in \bigwedge^1 \otimes E$ with $\mathrm{d}^\nabla \psi \in \mathrm{rg}(R)$, there is an $\eta \in E$ such that $\psi = \nabla\eta$.
- Clearly, if $\mathrm{d}^\nabla : \bigwedge^1 \otimes E \longrightarrow \bigwedge^2 \otimes E$ is injective, then ∇ is exact,

$$\mathrm{d}^\nabla \psi = R(\eta) \quad \Longrightarrow \quad \mathrm{d}^\nabla(\psi - \nabla\eta) = 0.$$

- We call ∇ *generic*, if $R : \bigwedge^1 \otimes E \to \bigwedge^3 \otimes E$ is injective.

Lemma

If ∇ is generic, then ∇ is exact and $R : E \to \bigwedge^2 \otimes E$ is injective.

The curvature filtration

How can we use genericity to show a connection is exact, even though the curvature $R : E \to \wedge^2 \otimes E$ is not injective?
Define $E_0 := \ker(R : E \to \wedge^2 \otimes E)$ and consider $E/E_0 \to \wedge^2 \otimes E/E_0$, however, the induced map may still be not injective $\rightsquigarrow$ curvature filtration of E:

$$E_0 \subset E_1 \subset \ldots \subset \ E_k := \{\eta \in E \mid R(\eta) \in \wedge^2 \otimes E_{k-1}\} \subset E_{k+1} \subset \ldots \subset E$$

Let $m = \min\{k \mid E_k = E_{k+1}\}$. Then $R : E/E_m \to \wedge^2 \otimes E/E_m$ is injective.

Proposition (CELM '22)

Let (E, ∇) be a vector bundle with connection. If

- *each E_k in the curvature filtration is ∇-parallel and*
- *the induced connection on E/E_m is exact,*

then (E, ∇) is exact. In particular, if the induced connection on E/E_m is generic, then (E, ∇) is exact.

Proof

- Assume that $\mathrm{d}^{\nabla}(\psi) = R(\eta)$. Wlog, we can assume that $\mathrm{d}^{\nabla}\psi = 0$.
- Since ∇ on E/E_m is exact with injective curvature,

$$\mathrm{d}^{\nabla}\psi = 0 \implies \psi = 0 \in \wedge^1 \otimes E/E_m, \text{ i.e. } \psi \in \wedge^1 \otimes E_m.$$

- Observe that each E_k/E_{k-1} is flat, and hence for each k,

$$\mathrm{d}^{\nabla} : \ker\left(\wedge^1 \otimes E_k/E_{k-1} \xrightarrow{\mathrm{d}^{\nabla}} \wedge^2 \otimes E_k/E_{k-1}\right) \longrightarrow \wedge^2 \otimes E_{k-1}/E_{k-2}$$

  is injective.
- This enables an induction $\rightsquigarrow \psi \in E_0$ with $\mathrm{d}^{\nabla}\psi = 0$.
- But E/E_0 is flat, so $\psi = \nabla\eta$.

The Killing connection

On a semi-Riemannian manifold, for $\xi \in \bigwedge^1$ and $h \in \bigodot^2$,

- ξ is Killing $\iff \nabla\xi = \mu \in \bigwedge^2$. This implies $\nabla_X \mu = R(\xi, X)$.
- $h = \mathcal{K}(\xi) \iff h = \nabla\xi - \mu$ with $\mu \in \bigwedge^2$. This implies

$$\mathrm{d}^\nabla h = \nabla\mu - R(\xi, -).$$

$\rightsquigarrow$ Killing connection $\overline{\nabla}$ on

$$E = \begin{matrix} \bigwedge^1 \\ \oplus \\ \bigwedge^2 \end{matrix} \ni \begin{pmatrix} \xi \\ \mu \end{pmatrix} \overset{\overline{\nabla}}{\longmapsto} \begin{pmatrix} \nabla\xi - \mu \\ \nabla\mu - R(\xi, -) \end{pmatrix} \in \bigwedge^1 \otimes E$$

- ξ is Killing $\iff \begin{pmatrix} \xi \\ \mathrm{d}\xi \end{pmatrix}$ is parallel for $\overline{\nabla}$, [Kostant '55].
- $h \in \mathrm{rg}(\mathcal{K}) \iff \begin{pmatrix} h \\ \mathrm{d}^\nabla h \end{pmatrix} \in \mathrm{rg}(\overline{\nabla}) \cap \begin{pmatrix} \bigodot^2 \\ \oplus \\ \bigwedge^1 \otimes \bigwedge^2 \end{pmatrix}$.

The Killing connection not only controls the Killing vector fields but also the range of the Killing operator!

The curvature of Killing connection

Let $\overline{R} \in \wedge^2 \otimes E$ be the curvature of the Killing connection,

$$\overline{R}\begin{pmatrix}\xi\\ \mu\end{pmatrix} = \begin{pmatrix}0\\ \mathcal{R}(\mu) - \nabla_\xi R\end{pmatrix},$$

i.e. the Killing connection is flat $\iff$ (M, g) has constant sectional curvature.

On the other hand, for $\begin{pmatrix}h\\ \mathrm{d}^\nabla h\end{pmatrix} \in \wedge^1 \otimes E$,

$$\mathrm{d}^{\overline{\nabla}}\begin{pmatrix}h\\ \mathrm{d}^\nabla h\end{pmatrix} = \begin{pmatrix}0\\ \mathcal{C}(h)\end{pmatrix}$$

Hence, if $\begin{pmatrix}h\\ \mathrm{d}^\nabla h\end{pmatrix} \in \mathrm{rg}(\overline{\nabla})$, we recover the fundamental equality

$$\mathcal{C} \circ \mathcal{K}(\xi) \;=\; \mathcal{R}(\mathrm{d}\xi) - \nabla_\xi R.$$

Proposition (CELM '22)

On a locally symmetric space, the Killing connection is exact $\iff$ $\wedge^1 \xrightarrow{\mathcal{K}} \odot^2 \xrightarrow{\widehat{\mathcal{C}}} \boxplus$ *is exact.*

Proof: $h \in \ker(\widehat{\mathcal{C}}) \iff \mathcal{C}(h) = \mathcal{R}(\mu) \iff \mathrm{d}^{\nabla}\begin{pmatrix} h \\ \mathrm{d}^{\nabla} h \end{pmatrix} \in \mathrm{rg}(\overline{R})$ with $h \in \odot^2$.

($\Rightarrow$) $\overline{\nabla}$ exact $\implies \exists\, \eta = \begin{pmatrix} \xi \\ \mu \end{pmatrix} \in E$, with $\psi = \overline{\nabla}\eta$ and hence $K(\xi) = h$.

($\Leftarrow$) Let $\psi = \begin{pmatrix} h+\omega \\ \phi \end{pmatrix} \in \wedge^1 \otimes E$ with $h \in \odot^2$ and $\omega \in \wedge^2$, such that $\mathrm{d}^{\nabla}(\psi) = \begin{pmatrix} 0 \\ \mathcal{R}(\mu) \end{pmatrix} \in \mathrm{rg}(\overline{R})$, then

$$\hat{\psi} = \psi + \overline{\nabla}\begin{pmatrix} 0 \\ \omega \end{pmatrix} = \begin{pmatrix} h \\ \hat{\phi} \end{pmatrix} \text{ and } \mathrm{d}^{\nabla}\hat{\psi} = \begin{pmatrix} 0 \\ \mathcal{R}(\mu+\omega) \end{pmatrix} \in \mathrm{rg}(\overline{R}).$$

This forces $\hat{\phi} = \mathrm{d}^{\nabla} h$ and hence $\mathcal{C}(h) = \mathcal{R}(\mu+\omega)$.

$\wedge^1 \xrightarrow{\mathcal{K}} \odot^2 \xrightarrow{\widehat{\mathcal{C}}} \boxplus$ exact $\implies h = \mathcal{K}(\xi)$ and hence $\hat{\psi} = \overline{\nabla}\begin{pmatrix} \xi \\ \mathrm{d}\xi \end{pmatrix}$.

The curvature filtration for the Killing connection

Let $\overline{R} \in \wedge^2 \otimes E$ the curvature of the Killing connection on a locally symmetric space,

$$\begin{pmatrix} \xi \\ A \end{pmatrix} \overset{\overline{\nabla}}{\longmapsto} \begin{pmatrix} \nabla\xi - A \\ \nabla A - R(\xi, -) \end{pmatrix}, \qquad \overline{R}\begin{pmatrix} \xi \\ A \end{pmatrix} = \begin{pmatrix} 0 \\ -A \cdot R \end{pmatrix}, \quad \text{for} \quad \begin{matrix} \xi \in TM, \\ A \in \mathfrak{so}(TM). \end{matrix}$$

- $E_0 = TM \oplus \mathfrak{h}_0$, where $\mathfrak{h}_0 = \mathfrak{aut}(R) = \{A \in \mathfrak{so}(TM) \mid A \cdot R = 0\}$.
  In particular, $\mathfrak{hol} = \mathrm{span}\{R(X, Y) \mid X, Y \in TM\} \subset \mathfrak{h}_0$.
- $E_k = TM \oplus \mathfrak{h}_k$, where $\mathfrak{h}_k = \{A \in \mathfrak{so}(TM) \mid [A, \mathfrak{hol}] \subset \mathfrak{h}_{k-1}\}$.
- $\nabla R = 0 \implies \nabla_X(A \cdot R) = (\nabla_X A) \cdot R \implies$
  E_0 and all other E_k's are parallel subbundles of E, so the Killing connection is exact if the induced connection on $\mathfrak{so}(TM)/\mathfrak{h}_m$ is exact.
- induced connection on $E/E_m = \mathfrak{so}(TM)/\mathfrak{h}_m$ is the Levi-Civita connection.

What is $\mathfrak{h}_m$ and is the Levi-Civita connection on $\mathfrak{so}(TM)/\mathfrak{h}_m$ exact?

De Rham decomposition

If (M, g) is a Riemannian or Lorentzian locally symmetric space, then its local de Rham decomposition is as follows:

$$(M, g) = (L, h) \times (N_1, g_1) \times \ldots \times (N_k, g_k),$$

- (N_i, g_i) are irreducible Riemannian locally symmetric spaces (or empty), and
- (L, h) is either flat (Euclidean or Minkowksi) or an indecomposable Lorentzian manifold, i.e. either (L, h)
  - has constant section curvature, i.e. locally de Sitter or anti de Sitter, or
  - is a Cahen-Wallach space.

Note that even though the tangent bundle of products decomposes into parallel subbundles, this is not the case for the prolongation connection.

Riemannian symmetric spaces

- If $\mathfrak{so}(TM) = \mathfrak{h}_0 \oplus \mathfrak{h}_0^\perp$, e.g. if g is Riemannian, then $\left[\mathfrak{h}_0, \mathfrak{h}_0^\perp\right] \subset \mathfrak{h}_0^\perp$, and hence $\mathfrak{h}_1 = \mathfrak{h}_0$, so $E/E_m = E/E_0 = \mathfrak{h}_0^\perp$.
- We show that $(\mathfrak{h}_0^\perp, \nabla)$ is exact by showing that it is generic, i.e. that

$$\begin{array}{ccccc} \wedge^1 \otimes \mathfrak{h}_0^\perp & \overset{R=(\mathrm{d}^\nabla)^2}{\longrightarrow} & \wedge^3 \otimes \mathfrak{h}_0^\perp & \text{is injective} & \\ \psi & \longmapsto & R(\psi), & R(\psi)(X,Y,Z) = \underset{X,Y,Z}{\mathrm{cyc}} \big(\big[\underbrace{R(X,Y)}_{\in \mathfrak{hol}}, \underbrace{\psi(Z)}_{\in \mathfrak{h}_0^\perp} \big] \big). & \end{array}$$

- If (M, g) is irreducible Riemannian symmetric with $Ric = \pm g$, then:
  $\psi \in \ker(R : \wedge^1 \otimes \mathfrak{h}_0^\perp \to \wedge^3 \otimes \mathfrak{h}_0^\perp)$
  $\Longrightarrow$ $\hat{\psi}_{abc} = \psi_{[ab]c}$ is tracefree and of the form

$$\hat{\psi} = \sum_i \omega_i \otimes \phi_i, \qquad \text{with } \omega_i \in \wedge^2 \text{ eigen-forms of } R : \wedge^2 \to \wedge^2 \text{ for eigenvalue 2.}$$

For an irreducible Riemannian symmetric space, what are the eigenvalues of

$$R : \wedge^2 \to \wedge^2, \qquad \omega_{ab} \mapsto R_{ab}{}^{cd} \omega_{cd} \quad ?$$

The eigenvalues of the curvature operator on 2-forms

Proposition (CELM '21)

Let (M, g) be an irreducible Riemannian locally symmetric space with $Ric = \pm g$. The the eigenvalues for $R : \wedge^2 \to \wedge^2$ lie between 0 and ± 2 and

- $\ker(R) = \mathfrak{h}_0^{\perp}$, *and*
- $R(\omega) = \pm 2\,\omega \iff (M, g)$ *is Kähler with Kähler form* ω.

Hence $\hat{\psi} = \sum_i \omega_i \otimes \phi_i = \omega \otimes \phi$, where ω is the Kähler form. Since $\hat{\psi}$ is trace free,

$$\omega(\phi, -) = 0,$$

which implies $\hat{\psi} = 0$ and $\psi = 0$, so $R : \wedge^1 \otimes \mathfrak{h}_0^{\perp} \to \wedge^3 \otimes \mathfrak{h}_0^{\perp}$ is injective.

Corollary

For a Riemannian irreducible locally symmetric space, the Levi-Civita connection on $\mathfrak{aut}(R)^{\perp}$ is exact and so is the Killing connection.

Indecomposable Lorentzian symmetric spaces

- Cahen & Wallach, '70: An indecomposable Lorentzian symmetric spaces either has *constant sectional curvature* or is universally covered by a *Cahen-Wallach space*.
- Constant sectional curvature (de Sitter or anti de Sitter space): the Killing connection is flat and hence exact. ✓
- *Cahen-Wallach space:* $M = \mathbb{R}^{n+2}$ with Lorentzian metric

$$g \;=\; 2\,\mathrm{d}v\,\mathrm{d}t + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j + Q_{ij}x^ix^j\,(\mathrm{d}t)^2,$$

where $(t, x^1, \ldots, x^n, v)$ are coordinates on $\mathbb{R}^{n+2}$ and $\det(Q_{ij}) \neq 0$.
The null vector field ∂_v is parallel and $\mathfrak{hol} =$

$$\underset{i=1,\ldots,n}{\operatorname{span}} \left\{ R(\partial_t, \partial_i) = \begin{pmatrix} 0 & -Q_{ij} & 0 \\ 0 & 0 & Q_i{}^j \\ 0 & 0 & 0 \end{pmatrix} \right\} \;\subset\; \left\{ \begin{pmatrix} -a & -u^\top & 0 \\ -w & B & u \\ 0 & w^\top & a \end{pmatrix} \;\middle|\; \begin{array}{l} u, w \in \mathbb{R}^n \\ A \in \mathfrak{so}(n) \\ a \in \mathbb{R} \end{array} \right\}$$

$$= \mathbb{R}^n \qquad\qquad = \; \mathfrak{so}(1, n+1).$$

The curvature filtration for the Killing connection of Cahen-Wallach spaces

Recall

$$E_0 = \bigwedge^1 \oplus \mathfrak{aut}(R) = \{A \in \mathfrak{so}(1, n+1) \mid [A, R(X,Y)] - R(AX, Y) - R(X, AY) = 0\}$$

- $\mathfrak{h}_0 = \mathfrak{aut}(R) = Z_{\mathfrak{so}(n)}(Q) \ltimes \mathbb{R}^n$
- $E_k = TM \oplus \mathfrak{h}_k$, where $\mathfrak{h}_k = \{A \in \mathfrak{so}(TM) \mid [A, \mathfrak{hol}] \subset \mathfrak{h}_{k-1}\}$.

$$\mathfrak{h}_1 = \left\{ \begin{pmatrix} a & u^t & 0 \\ 0 & B & -u \\ 0 & 0 & -a \end{pmatrix} \mid \begin{array}{l} a \in \mathbb{R}, \\ u \in \mathbb{R}^n, \\ B \in \mathfrak{so}(n) \end{array} \right\} = \mathfrak{co}(n) \ltimes \mathbb{R}^n$$

$$\mathfrak{h}_2 = \mathfrak{so}(1, n+1).$$

- $E_2 = E$ and $E/E_2 = \{0\}$ and hence the Killing connection for Cahen-Wallach spaces is exact ✓

How to deal with products?

Let $(M, g) = (M^1, g^1) \times (M^2, g^2)$ and denote $T^i = TM^i$, etc.

$$E = \begin{matrix} TM \\ \oplus \\ \mathfrak{so}(TM) \end{matrix} = \begin{matrix} T^1 \\ \oplus \\ \mathfrak{so}(T^1) \end{matrix} \oplus \begin{matrix} T^2 \\ \oplus \\ \mathfrak{so}(T^2) \end{matrix} \oplus T^1 \wedge T^2.$$

- If R^1 is injective, which we assume from now on, then

$$E/E_m = \mathfrak{so}(T^1)/\mathfrak{h}^1_m \oplus \mathfrak{so}(T^2)/\mathfrak{h}^2_m \oplus (T^1 \wedge T^2),$$

  and this decomposition is parallel for ∇. Hence, the Killing connection of (M, g) is exact if each of the three summands is exact. We have exactness for $\mathfrak{so}(T^i)/\mathfrak{h}^i_m$, so it remains to show that ∇ is exact on $T^1 \wedge T^2$.
- Again we show genericity of ∇ on $T^1 \wedge T^2$: If $\psi \in \ker((\mathrm{d}^\nabla)^2)$, then $\psi \in T^*M \otimes (T^1 \wedge T^2)$ and $\psi_{[ab]c}$ is of the form

$$\omega \wedge \phi, \quad \text{with } \omega \text{ a 2-form on } M^1 \text{ and } \phi \text{ a parallel 1-form on } M^2$$

  such that

$$R^1(\omega) = \pm 2\omega, \quad \text{i.e. } \omega \text{ is the Kähler form.}$$

  That's why exactness does not hold for Kähler $\times$ flat.

The counterexample

Let $(M, g) = (M_1, g_1) \times (M_2, g_2)$ with

- (M_1, g_1) irreducible Hermitian symmetric with Kählerform ω and Kähler potential ϕ, and
- (M_2, g_2) locally symmetric with parallel vector field ξ.

Then, $h := \phi \cdot \xi \in \odot^2$,

$$\mathcal{C}(h) \;=\; \mathcal{R}(\phi \wedge \xi),$$

and hence h is in the kernel of $\widehat{\mathcal{C}}$. But h is not in the range of $\mathcal{K}$.

Happy Birthday, Christian!