Mathematical reflections on locality

Sylvie Paycha
University of Potsdam
joint work with Li Guo and Bin Zhang

Freiburg, on the occasion of Christian Bär's 60th birthday,
September 20th 2022

The Dirac operator attacking a Lorentzian manifold

The Dirac operator attacking a Lorentzian manifold

JOYEUX ANNIVERSAIRE CHRISTIAN!

I. The Locality principle revisited

Locality principle

The principle of locality (or locality principle) states that an object is influenced directly only by its immediate surroundings.

Locality principle

The principle of locality (or locality principle) states that an object is influenced directly only by its immediate surroundings.

Thus, one can separate events located in different regions of space-time and should be able to measure them independently.

Locality principle

The principle of locality (or locality principle) states that an object is influenced directly only by its immediate surroundings.

Thus, one can separate events located in different regions of space-time and should be able to measure them independently.

Our aim

- Propose a mathematical framework which encompasses the main features of the locality principle in QFT;

Locality principle

The principle of locality (or locality principle) states that an object is influenced directly only by its immediate surroundings.

Thus, one can separate events located in different regions of space-time and should be able to measure them independently.

Our aim

- Propose a mathematical framework which encompasses the main features of the locality principle in QFT;
- use this framework to carry out renormalisation in accordance with the locality principle.

Causal separation

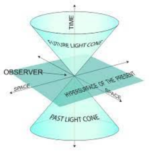
Light cone, past and future

In the Minkowski space $(\mathbb{R}^d, g)$, where $g(x,y) = -x_0 y_0 + \sum_{j=1}^{d-1} x_j y_j$ is the Lorentzian scalar product,

Causal separation

Light cone, past and future

In the Minkowski space $(\mathbb{R}^d, g)$, where $g(x,y) = -x_0 y_0 + \sum_{j=1}^{d-1} x_j y_j$ is the Lorentzian scalar product, there is a notion of "past" and "future":

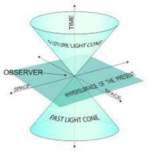


(picture downloaded from Wikipedia)

Causal separation

Light cone, past and future

In the Minkowski space $(\mathbb{R}^d, g)$, where $g(x,y) = -x_0 y_0 + \sum_{j=1}^{d-1} x_j y_j$ is the Lorentzian scalar product, there is a notion of "past" and "future":



(picture downloaded from Wikipedia)

Two sets S_1 and S_2 are causally separated $(S_1 || S_2)$ if and only if S_i does not lie in the future of S_j for $i \neq j$.

Locality in axiomatic QFT

The Wightman field $\varphi : \mathcal{S}(\mathbb{R}^d) \to \mathcal{O}(H)$ obeys the locality axiom

$$\mathrm{Supp}(f_1) \| \mathrm{Supp}(f_2) \Longrightarrow [\varphi(f_1), \varphi(f_2)] = 0. \tag{1}$$

Locality in axiomatic QFT

The Wightman field $\varphi : \mathcal{S}(\mathbb{R}^d) \to \mathcal{O}(H)$ obeys the locality axiom

$$\mathrm{Supp}(f_1) \| \mathrm{Supp}(f_2) \Longrightarrow [\varphi(f_1), \varphi(f_2)] = 0. \tag{1}$$

The (relative) scattering matrix S_f satisfies the locality condition

$$\mathrm{Supp}(f_1) \| \mathrm{Supp}(f_2) \quad \Longrightarrow \quad S_f(f_1 + f_2) = S_f(f_1)\, S_f(f_2)$$

Locality in axiomatic QFT

The Wightman field $\varphi : \mathcal{S}(\mathbb{R}^d) \to \mathcal{O}(H)$ obeys the locality axiom

$$\mathrm{Supp}(f_1) \| \mathrm{Supp}(f_2) \Longrightarrow [\varphi(f_1), \varphi(f_2)] = 0. \tag{1}$$

The (relative) scattering matrix S_f satisfies the locality condition

$$\begin{aligned} \mathrm{Supp}(f_1) \| \mathrm{Supp}(f_2) \quad &\Longrightarrow \quad S_f(f_1 + f_2) = S_f(f_1)\, S_f(f_2) \\ &\Longrightarrow \quad [S_f(f_1), S_f(f_2)] = 0. \end{aligned} \tag{2}$$

Mathematical interpretation

We introduce two binary relations

- on sets:

$$O_1 \top' O_2 :\Leftrightarrow [O_1, O_2] = 0, \tag{3}$$

Mathematical interpretation

We introduce two binary relations

- on sets:

$$O_1 \top' O_2 :\Leftrightarrow [O_1, O_2] = 0, \tag{3}$$

- on test functions:

$$f_1 \top f_2 :\Leftrightarrow \mathrm{Supp}(f_1) \| \mathrm{Supp}(f_2). \tag{4}$$

Mathematical interpretation

We introduce two binary relations

- on sets:

$$O_1 \top' O_2 :\Leftrightarrow [O_1, O_2] = 0, \tag{3}$$

- on test functions:

$$f_1 \top f_2 :\Leftrightarrow \mathrm{Supp}(f_1) \| \mathrm{Supp}(f_2). \tag{4}$$

Interpretation of (1)

$$f_1 \top f_2 \Longrightarrow \varphi(f_1) \top' \varphi(f_2). \tag{5}$$

Mathematical interpretation

We introduce two binary relations

- on sets:

$$O_1 \top' O_2 :\Leftrightarrow [O_1, O_2] = 0, \tag{3}$$

- on test functions:

$$f_1 \top f_2 :\Leftrightarrow \mathrm{Supp}(f_1) \| \mathrm{Supp}(f_2). \tag{4}$$

Interpretation of (1)

$$f_1 \top f_2 \Longrightarrow \varphi(f_1) \top' \varphi(f_2). \tag{5}$$

Interpretation of (2)

$$f_1 \top f_2 \Longrightarrow S_f(f_1 + f_2) = S_f(f_1)\, S_f(f_2). \tag{6}$$

II. Locality as a symmetric binary relation

Algebraic locality

Definition of locality

A **locality set** is a couple $(X, \top)$ where X is a set and $\top \subseteq X \times X$ is a symmetric relation on X, called **locality relation** (or **independence relation**) of the locality set:

Algebraic locality

Definition of locality

A **locality set** is a couple $(X, \top)$ where X is a set and $\top \subseteq X \times X$ is a symmetric relation on X, called **locality relation** (or **independence relation**) of the locality set:

$$x_1 \top x_2 \Longleftrightarrow (x_1, x_2) \in \top, \quad \forall x_1, x_2 \in X.$$

Algebraic locality

Definition of locality

A **locality set** is a couple $(X, \top)$ where X is a set and $\top \subseteq X \times X$ is a symmetric relation on X, called **locality relation** (or **independence relation**) of the locality set:

$$x_1 \top x_2 \Longleftrightarrow (x_1, x_2) \in \top, \quad \forall x_1, x_2 \in X.$$

First examples of locality

- $X \top_\cap Y \Longleftrightarrow X \cap Y = \emptyset$ on subsets X, Y of a set Z.

Algebraic locality

Definition of locality

A **locality set** is a couple $(X, \top)$ where X is a set and $\top \subseteq X \times X$ is a symmetric relation on X, called **locality relation** (or **independence relation**) of the locality set:

$$x_1 \top x_2 \iff (x_1, x_2) \in \top, \quad \forall x_1, x_2 \in X.$$

First examples of locality

- $X \top_\cap Y \iff X \cap Y = \emptyset$ on subsets X, Y of a set Z.
- $X \top Y \iff X \perp Y$ on subsets X, Y of an euclidean vector space $(V, \perp)$.

Algebraic locality

Definition of locality

A **locality set** is a couple $(X, \top)$ where X is a set and $\top \subseteq X \times X$ is a symmetric relation on X, called **locality relation** (or **independence relation**) of the locality set:

$$x_1 \top x_2 \iff (x_1, x_2) \in \top, \quad \forall x_1, x_2 \in X.$$

First examples of locality

- $X \top_\cap Y \iff X \cap Y = \emptyset$ on subsets X, Y of a set Z.
- $X \top Y \iff X \perp Y$ on subsets X, Y of an euclidean vector space $(V, \perp)$.

(almost-)Separation of supports

Let $U \subset \mathbb{R}^n$ be an open subset and $\epsilon \geq 0$. Two functions ϕ, ψ in $\mathcal{D}(U)$ are independent i.e., $\phi \top_\epsilon \psi$ whenever

Algebraic locality

Definition of locality

A **locality set** is a couple $(X, \top)$ where X is a set and $\top \subseteq X \times X$ is a symmetric relation on X, called **locality relation** (or **independence relation**) of the locality set:

$$x_1 \top x_2 \Longleftrightarrow (x_1, x_2) \in \top, \quad \forall x_1, x_2 \in X.$$

First examples of locality

- $X \top_\cap Y \Longleftrightarrow X \cap Y = \emptyset$ on subsets X, Y of a set Z.
- $X \top Y \Longleftrightarrow X \perp Y$ on subsets X, Y of an euclidean vector space $(V, \perp)$.

(almost-)Separation of supports

Let $U \subset \mathbb{R}^n$ be an open subset and $\epsilon \geq 0$. Two functions ϕ, ψ in $\mathcal{D}(U)$ are independent i.e., $\phi \top_\epsilon \psi$ whenever $d\left(\mathrm{Supp}(\phi), \mathrm{Supp}(\psi)\right) > \epsilon$.

Algebraic locality

Definition of locality

A **locality set** is a couple $(X, \top)$ where X is a set and $\top \subseteq X \times X$ is a symmetric relation on X, called **locality relation** (or **independence relation**) of the locality set:

$$x_1 \top x_2 \Longleftrightarrow (x_1, x_2) \in \top, \quad \forall x_1, x_2 \in X.$$

First examples of locality

- $X \top_\cap Y \Longleftrightarrow X \cap Y = \emptyset$ on subsets X, Y of a set Z.
- $X \top Y \Longleftrightarrow X \perp Y$ on subsets X, Y of an euclidean vector space $(V, \perp)$.

(almost-)Separation of supports

Let $U \subset \mathbb{R}^n$ be an open subset and $\epsilon \geq 0$. Two functions ϕ, ψ in $\mathcal{D}(U)$ are independent i.e., $\phi \top_\epsilon \psi$ whenever $d\,(\mathrm{Supp}(\phi), \mathrm{Supp}(\psi)) > \epsilon$.

For $\epsilon = 0$,

Algebraic locality

Definition of locality

A **locality set** is a couple $(X, \top)$ where X is a set and $\top \subseteq X \times X$ is a symmetric relation on X, called **locality relation** (or **independence relation**) of the locality set:

$$x_1 \top x_2 \Longleftrightarrow (x_1, x_2) \in \top, \quad \forall x_1, x_2 \in X.$$

First examples of locality

- $X \top_\cap Y \Longleftrightarrow X \cap Y = \emptyset$ on subsets X, Y of a set Z.
- $X \top Y \Longleftrightarrow X \perp Y$ on subsets X, Y of an euclidean vector space $(V, \perp)$.

(almost-)Separation of supports

Let $U \subset \mathbb{R}^n$ be an open subset and $\epsilon \geq 0$. Two functions ϕ, ψ in $\mathcal{D}(U)$ are independent i.e., $\phi \top_\epsilon \psi$ whenever $d\left(\mathrm{Supp}(\phi), \mathrm{Supp}(\psi)\right) > \epsilon$.

For $\epsilon = 0$, this amounts to disjointness of supports, otherwise to ϵ-separation of supports.

Further examples

Probability theory: independence of events

Given a probability space $\mathcal{P} := (\Omega, \Sigma, P)$ and two events $A, B \in \Sigma$:

$$A \top B \iff P(A \cap B) = P(A)\, P(B).$$

Further examples

Probability theory: independence of events

Given a probability space $\mathcal{P} := (\Omega, \Sigma, P)$ and two events $A, B \in \Sigma$:

$$A \top B \iff P(A \cap B) = P(A)\, P(B).$$

Geometry: transversal manifolds

Given two submanifolds L_1 and L_2 of a manifold M:

$$L_1 \top L_2 \iff L_1 \pitchfork L_2 \iff T_x L_1 + T_x L_2 = T_x M \quad \forall x \in L_1 \cap L_2.$$

Further examples

Probability theory: independence of events

Given a probability space $\mathcal{P} := (\Omega, \Sigma, P)$ and two events $A, B \in \Sigma$:

$$A \top B \Longleftrightarrow P(A \cap B) = P(A)\, P(B).$$

Geometry: transversal manifolds

Given two submanifolds L_1 and L_2 of a manifold M:

$$L_1 \top L_2 \Longleftrightarrow L_1 \pitchfork L_2 \Longleftrightarrow T_x L_1 + T_x L_2 = T_x M \quad \forall x \in L_1 \cap L_2.$$

Number theory: coprime numbers

Given two positive integers m, n in $\mathbb{N}$:

$$m \top n \Longleftrightarrow m \wedge n = 1.$$

Locality category

Locality structures

- set $X \rightsquigarrow$ locality set $(X, \top)$;

Locality category

Locality structures

- set X $\rightsquigarrow$ locality set $(X, \top)$;
- semi-group (X, m_X) $\rightsquigarrow$ locality semi-group $(X, m_X, \top)$

  $(U \subset V \Longrightarrow U^\top$ semi-group$)$;

Locality category

Locality structures

- set $X \rightsquigarrow$ locality set $(X, \top)$;
- semi-group $(X, m_X) \rightsquigarrow$ locality semi-group $(X, m_X, \top)$

  $(U \subset V \Longrightarrow U^\top$ semi-group$)$;
- vector space $(V, +, \cdot) \rightsquigarrow$ locality vector space $(V, +, \cdot, \top)$

  $(U \subset V \Longrightarrow U^\top$ vector space$)$;

Locality category

Locality structures

- set $X \rightsquigarrow$ locality set $(X, \top)$;
- semi-group $(X, m_X) \rightsquigarrow$ locality semi-group $(X, m_X, \top)$
  $(U \subset V \Longrightarrow U^\top$ semi-group$)$;
- vector space $(V, +, \cdot) \rightsquigarrow$ locality vector space $(V, +, \cdot, \top)$
  $(U \subset V \Longrightarrow U^\top$ vector space$)$;
- algebra $(A, +, \cdot, m_A) \rightsquigarrow$ locality algebra $(A, +, \cdot, m_A, \top)$.

Locality morphisms: $f : (X, \top_X) \to (Y, \top_Y)$

- locality map:
  $(f \times f)(\top_X) \subset \top_Y$ or equivalently $x_1 \top_X x_2 \Longrightarrow f(x_1) \top_Y f(x_2)$;

Locality category

Locality structures

- set $X \rightsquigarrow$ locality set $(X, \top)$;
- semi-group $(X, m_X) \rightsquigarrow$ locality semi-group $(X, m_X, \top)$
  $(U \subset V \Longrightarrow U^\top$ semi-group$)$;
- vector space $(V, +, \cdot) \rightsquigarrow$ locality vector space $(V, +, \cdot, \top)$
  $(U \subset V \Longrightarrow U^\top$ vector space$)$;
- algebra $(A, +, \cdot, m_A) \rightsquigarrow$ locality algebra $(A, +, \cdot, m_A, \top)$.

Locality morphisms: $f : (X, \top_X) \to (Y, \top_Y)$

- locality map:
  $(f \times f)(\top_X) \subset \top_Y$ or equivalently $x_1 \top_X x_2 \Longrightarrow f(x_1) \top_Y f(x_2)$;
- locality semi-group morphism $f : (X, m_X, \top_X) \to (Y, m_Y, \top_Y)$:
  f is a locality map and $x_1 \top_X x_2 \Longrightarrow f(m_X(x_1, x_2)) = m_Y(f(x_1), f(x_2))$...

III. Locality relations are ubiquous

Local functionals

are functionals F on test functions (fields) φ of the form $F(\varphi) = \int_M f\left(j_x^k(\varphi)\right)\, dx$ (here $j_x^k(\phi)$ is the k-th jet of ϕ at x): The localised version at φ:

$$F(\varphi + \psi) = F(\varphi) + \int_M f\left(j_x^k(\psi)\right)\, dx \quad \forall \psi. \tag{7}$$

Local functionals

are functionals F on test functions (fields) φ of the form $F(\varphi) = \int_M f\left(j_x^k(\varphi)\right)\, dx$ (here $j_x^k(\phi)$ is the k-th jet of ϕ at x): The localised version at φ:

$$F(\varphi+\psi) = F(\varphi) + \int_M f\left(j_x^k(\psi)\right)\, dx \quad \forall\psi. \tag{7}$$

Hammerstein property/partial additivity similar to a causality condition on S-matrices of [Epstein, Glaser (1973)], [Bogoliubov, Shirkov (1959))], [Stückelberg (1950, 1951)]

$$\varphi_1 \top\!\!\cap \varphi_2 \Longrightarrow F(\varphi_1+\varphi+\varphi_2) = F(\varphi_1+\varphi) - F(\varphi) + F(\varphi+\varphi_2). \tag{8}$$

Local functionals

are functionals F on test functions (fields) φ of the form $F(\varphi) = \int_M f\left(j_x^k(\varphi)\right)\, dx$ (here $j_x^k(\phi)$ is the k-th jet of ϕ at x): The localised version at φ:

$$F(\varphi+\psi) = F(\varphi) + \int_M f\left(j_x^k(\psi)\right)\, dx \quad \forall \psi. \tag{7}$$

Hammerstein property/partial additivity similar to a causality condition on S-matrices of [Epstein, Glaser (1973)], [Bogoliubov, Shirkov (1959))], [Stückelberg (1950, 1951)]

$$\varphi_1 \top_\cap \varphi_2 \Longrightarrow F(\varphi_1+\varphi+\varphi_2) = F(\varphi_1+\varphi) - F(\varphi) + F(\varphi+\varphi_2). \tag{8}$$

Comparing the two [Brouder, Dang, Laurent-Gengoux, Rejzner (2018)]
Provided $D_\varphi F$ can be represented as a function $\nabla_\varphi F$ such that the map $\varphi \mapsto \nabla_\varphi F$ is smooth, then (8) $\Longleftrightarrow$ (7).

Local linear forms on pseudodifferential operators

$\Psi^{\Gamma}_{\mathrm{phg}}(M)$ polyhomog. pseudodiff. operators on M with order in $\Gamma \subset \mathbb{C}$.

Locality of linear forms

A linear form $\Lambda : \Psi^{\Gamma}_{\mathrm{phg}}(M) \longrightarrow \mathbb{C}$ is local if and only if

$$\chi \stackrel{\top}{\cap} \chi' \Longrightarrow \Lambda(\chi\, A\, \chi') = 0.$$

Local linear forms on pseudodifferential operators

$\Psi^{\Gamma}_{\mathrm{phg}}(M)$ polyhomog. pseudodiff. operators on M with order in $\Gamma \subset \mathbb{C}$.

Locality of linear forms

A linear form $\Lambda : \Psi^{\Gamma}_{\mathrm{phg}}(M) \longrightarrow \mathbb{C}$ is local if and only if

$$\chi \top_{\cap} \chi' \Longrightarrow \Lambda(\chi\, A\, \chi') = 0.$$

A local linear form: Wodzicki residue

$\mathrm{res} : \Psi^{\mathbb{Z}}_{\mathrm{phg}}(M) \longrightarrow \mathbb{C}$ defined as an integral of the trace of the homogeneous component of the symbol of degree $-\dim$, is local.

Local linear forms on pseudodifferential operators

$\Psi^{\Gamma}_{\rm phg}(M)$ polyhomog. pseudodiff. operators on M with order in $\Gamma \subset \mathbb{C}$.

Locality of linear forms

A linear form $\Lambda : \Psi^{\Gamma}_{\rm phg}(M) \longrightarrow \mathbb{C}$ is local if and only if

$$\chi \top_{\cap} \chi' \Longrightarrow \Lambda(\chi\, A\, \chi') = 0.$$

A local linear form: Wodzicki residue

res : $\Psi^{\mathbb{Z}}_{\rm phg}(M) \longrightarrow \mathbb{C}$ defined as an integral of the trace of the homogeneous component of the symbol of degree $-$ dim, is local.

The index as a Wodzicki residue

The index of a differential operator D of Dirac-type is local since

$$\mathrm{ind}(D) \sim \mathrm{res}(\log(D^2)).$$

Locality and singularities

Separation of wavefront sets

We define two locality relations on on $\mathcal{D}'(U)$, $U \subset \mathbb{R}^n$:

$$v_1 \top^{\mathrm{sing}} v_2 \iff \mathrm{Singsupp}(v_1) \cap \mathrm{Singsupp}(v_2) = \emptyset,$$

Locality and singularities

Separation of wavefront sets

We define two locality relations on on $\mathcal{D}'(U)$, $U \subset \mathbb{R}^n$:

$$v_1 \top^{\mathrm{sing}} v_2 \Longleftrightarrow \mathrm{Singsupp}(v_1) \cap \mathrm{Singsupp}(v_2) = \emptyset,$$

$$\text{and} \quad v_1 \top^{\mathrm{WF}} v_2 \Longleftrightarrow \mathrm{WF}(v_1) \cap \mathrm{WF}'(v_2) = \emptyset$$

where we have set $\mathrm{WF}'(v) := \{(x, -\xi) \in U \times (\mathbb{R}^n \setminus \{0\}) \mid (x, \xi) \in \mathrm{WF}(v)\}$.

Counterexample

Distributions can be independent for $\top^{\mathrm{WF}}$ and not for $\top^{\mathrm{sing}}$. We have $v_1 \top^{\mathrm{sing}} v_2 \Longrightarrow v_1 \top^{\mathrm{WF}} v_2$

Locality and singularities

Separation of wavefront sets

We define two locality relations on on $\mathcal{D}'(U)$, $U \subset \mathbb{R}^n$:

$$v_1 \top^{\text{sing}} v_2 \iff \mathrm{Singsupp}(v_1) \cap \mathrm{Singsupp}(v_2) = \emptyset,$$

$$\text{and} \quad v_1 \top^{\mathrm{WF}} v_2 \iff \mathrm{WF}(v_1) \cap \mathrm{WF}'(v_2) = \emptyset$$

where we have set $\mathrm{WF}'(v) := \{(x, -\xi) \in U \times (\mathbb{R}^n \setminus \{0\}) \mid (x, \xi) \in \mathrm{WF}(v)\}$.

Counterexample

Distributions can be independent for $\top^{\mathrm{WF}}$ and not for $\top^{\text{sing}}$. We have $v_1 \top^{\text{sing}} v_2 \Longrightarrow v_1 \top^{\mathrm{WF}} v_2$ but not conversely.

Locality and singularities

Separation of wavefront sets

We define two locality relations on on $\mathcal{D}'(U)$, $U \subset \mathbb{R}^n$:

$$v_1 \top^{\mathrm{sing}} v_2 \Longleftrightarrow \mathrm{Singsupp}(v_1) \cap \mathrm{Singsupp}(v_2) = \emptyset,$$

$$\text{and} \quad v_1 \top^{\mathrm{WF}} v_2 \Longleftrightarrow \mathrm{WF}(v_1) \cap \mathrm{WF}'(v_2) = \emptyset$$

where we have set $\mathrm{WF}'(v) := \{(x, -\xi) \in U \times (\mathbb{R}^n \setminus \{0\}) \,|\, (x, \xi) \in \mathrm{WF}(v)\}$.

Counterexample

Distributions can be independent for $\top^{\mathrm{WF}}$ and not for $\top^{\mathrm{sing}}$. We have $v_1 \top^{\mathrm{sing}} v_2 \Longrightarrow v_1 \top^{\mathrm{WF}} v_2$ but not conversely. The wavefront sets of $\nu_1(\phi) := \int_{\mathbb{R}^2} \phi(0, y)\, dy$ and $\nu_2(\phi) := \int_{\mathbb{R}^2} \phi(x, 0)\, dx$ read

$\mathrm{WF}(\nu_1) = \{((0, y); (\lambda, 0)) \,|\, y \in \mathbb{R}, \lambda \in \mathbb{R} \setminus \{0\}\}$; $\mathrm{WF}(\nu_2) = \{((x, 0); (0, \mu)) \,|\, x \in \mathbb{R}, \mu \in \mathbb{R} \setminus \{0\}\}$,

Locality and singularities

Separation of wavefront sets

We define two locality relations on on $\mathcal{D}'(U)$, $U \subset \mathbb{R}^n$:

$$v_1 \top^{\mathrm{sing}} v_2 \Longleftrightarrow \mathrm{Singsupp}(v_1) \cap \mathrm{Singsupp}(v_2) = \emptyset,$$

$$\text{and} \quad v_1 \top^{\mathrm{WF}} v_2 \Longleftrightarrow \mathrm{WF}(v_1) \cap \mathrm{WF}'(v_2) = \emptyset$$

where we have set $\mathrm{WF}'(v) := \{(x, -\xi) \in U \times (\mathbb{R}^n \setminus \{0\}) \,|\, (x, \xi) \in \mathrm{WF}(v)\}$.

Counterexample

Distributions can be independent for $\top^{\mathrm{WF}}$ and not for $\top^{\mathrm{sing}}$. We have $v_1 \top^{\mathrm{sing}} v_2 \Longrightarrow v_1 \top^{\mathrm{WF}} v_2$ but not conversely. The wavefront sets of $\nu_1(\phi) := \int_{\mathbb{R}^2} \phi(0, y)\, dy$ and $\nu_2(\phi) := \int_{\mathbb{R}^2} \phi(x, 0)\, dx$ read

$\mathrm{WF}(\nu_1) = \{((0, y); (\lambda, 0)) \,|\, y \in \mathbb{R}, \lambda \in \mathbb{R} \setminus \{0\}\}$; $\mathrm{WF}(\nu_2) = \{((x, 0); (0, \mu)) \,|\, x \in \mathbb{R}, \mu \in \mathbb{R} \setminus \{0\}\}$,

so $\nu_1 \top^{\mathrm{WF}} \nu_2$

Locality and singularities

Separation of wavefront sets

We define two locality relations on on $\mathcal{D}'(U)$, $U \subset \mathbb{R}^n$:

$$v_1 \top^{\mathrm{sing}} v_2 \Longleftrightarrow \mathrm{Singsupp}(v_1) \cap \mathrm{Singsupp}(v_2) = \emptyset,$$

$$\text{and} \quad v_1 \top^{\mathrm{WF}} v_2 \Longleftrightarrow \mathrm{WF}(v_1) \cap \mathrm{WF}'(v_2) = \emptyset$$

where we have set $\mathrm{WF}'(v) := \{(x, -\xi) \in U \times (\mathbb{R}^n \setminus \{0\}) \,|\, (x, \xi) \in \mathrm{WF}(v)\}$.

Counterexample

Distributions can be independent for $\top^{\mathrm{WF}}$ and not for $\top^{\mathrm{sing}}$. We have $v_1 \top^{\mathrm{sing}} v_2 \Longrightarrow v_1 \top^{\mathrm{WF}} v_2$ but not conversely. The wavefront sets of $\nu_1(\phi) := \int_{\mathbb{R}^2} \phi(0, y)\, dy$ and $\nu_2(\phi) := \int_{\mathbb{R}^2} \phi(x, 0)\, dx$ read

$\mathrm{WF}(\nu_1) = \{((0, y); (\lambda, 0)) \,|\, y \in \mathbb{R}, \lambda \in \mathbb{R} \setminus \{0\}\}$; $\mathrm{WF}(\nu_2) = \{((x, 0); (0, \mu)) \,|\, x \in \mathbb{R}, \mu \in \mathbb{R} \setminus \{0\}\}$,

so $\nu_1 \top^{\mathrm{WF}} \nu_2$ but $\nu_1 \not\top^{\mathrm{sing}} \nu_2$.

Partial product and locality

Partial product (Hörmander) of distributions

$\nu_1 \top^{\mathrm{WF}} \nu_2 \Rightarrow \big($the product $\nu_1 \cdot \nu_2$ is well-defined.$\big)$

Partial product and locality

Partial product (Hörmander) of distributions

$\nu_1 \top^{\mathrm{WF}} \nu_2 \Rightarrow$ (the product $\nu_1 \cdot \nu_2$ is well-defined.)

Partial product of pseudodifferential operators of non-integer order

We equip $\Psi^{\notin \mathbb{Z}}_{\mathrm{pgh}}$ (the canonical trace TR is well defined) with the locality relation
$A_1 \top^{\notin \mathbb{Z}} A_2 :\Leftrightarrow (\mathrm{ord}(A_1) + \mathrm{ord}(A_2) \notin \mathbb{Z}) \Rightarrow (\mathrm{TR}([A_1, A_2]) = 0)$.

Partial product and locality

Partial product (Hörmander) of distributions

$\nu_1 \top^{\mathrm{WF}} \nu_2 \Rightarrow \big($the product $\nu_1 \cdot \nu_2$ is well-defined.$\big)$

Partial product of pseudodifferential operators of non-integer order

We equip $\Psi^{\notin \mathbb{Z}}_{\mathrm{pgh}}$ (the canonical trace TR is well defined) with the locality relation
$A_1 \top^{\notin \mathbb{Z}} A_2 :\Leftrightarrow (\mathrm{ord}(A_1) + \mathrm{ord}(A_2) \notin \mathbb{Z}) \Rightarrow (\mathrm{TR}([A_1, A_2]) = 0)$.

Counterexample

Yet $\mathbb{C}$ equipped with the locality relation $x \top^{\notin \mathbb{Z}} y \Longleftrightarrow x + y \notin \mathbb{Z}$.
$(\mathbb{C}, \top, +)$ is NOT a locality semi-group:

Partial product and locality

Partial product (Hörmander) of distributions

$\nu_1 \top^{\mathrm{WF}} \nu_2 \Rightarrow$ (the product $\nu_1 \cdot \nu_2$ is well-defined.)

Partial product of pseudodifferential operators of non-integer order

We equip $\Psi_{\mathrm{pgh}}^{\notin\mathbb{Z}}$ (the canonical trace TR is well defined) with the locality relation
$A_1 \top^{\notin\mathbb{Z}} A_2 :\Leftrightarrow (\mathrm{ord}(A_1) + \mathrm{ord}(A_2) \notin \mathbb{Z}) \Rightarrow (\mathrm{TR}([A_1, A_2]) = 0)$.

Counterexample

Yet $\mathbb{C}$ equipped with the locality relation $x \top^{\notin\mathbb{Z}} y \iff x + y \notin \mathbb{Z}$.
$(\mathbb{C}, \top, +)$ is NOT a locality semi-group:for $U = \{1/3\}$ we have
$(1/3, 1/3) \in (U^\top \times U^\top) \cap \top$ but $1/3 + 1/3 = 2/3 \notin U^\top$.

IV. Evaluating meromorphic germs at poles in QFT

Functions of several variables in QFT

Speer's analytic renormalisation JMP 1967] revisited

Eugene Speer considers Feynman amplitudes given by the coefficients of the perturbation-series expansion of the S matrix in a Lagrangian field theory (with non zero mass).

Functions of several variables in QFT

Speer's analytic renormalisation JMP 1967] revisited

Eugene Speer considers Feynman amplitudes given by the coefficients of the perturbation-series expansion of the S matrix in a Lagrangian field theory (with non zero mass).

Excerpt of Speer's article

In this paper we apply a method of defining divergent quantities which was originated by Riesz and has been used in various contexts by many authors. [....] We find it necessary to consider functions of several complex variables $z_1, \cdots, z_k$*, one associated with each line of the Feynman graph. The main difficulty is the extension of the above [Riesz's] treatment of poles to the more complicated singularities which occur in several complex variables...*

Brain teaser

(We assume the poles are at zero)
Speer shows [Theorem 1] that the divergent expressions lie in the filtered algebra $\mathcal{M}^{\mathrm{Feyn}}(\mathbb{C}^\infty) := \cup_{k=1}^{\infty} \mathcal{M}^{\mathrm{Feyn}}(\mathbb{C}^k)$ consisting of Feynman functions $f : \mathbb{C}^k \to \mathbb{C}$,

Brain teaser

(We assume the poles are at zero)
Speer shows [Theorem 1] that the divergent expressions lie in the filtered algebra $\mathcal{M}^{\mathrm{Feyn}}(\mathbb{C}^\infty) := \cup_{k=1}^{\infty}\mathcal{M}^{\mathrm{Feyn}}(\mathbb{C}^k)$ consisting of Feynman functions $f : \mathbb{C}^k \to \mathbb{C}$,

$$f = \frac{h(z_1, \cdots, z_k)}{{L_1}^{s_1} \cdots {L_m}^{s_m}}, \quad L_i = \sum_{j \in J_i} z_j, \quad J_i \subset \{1, \cdots, k\}, \ h \text{ holom. at zero.}$$

Brain teaser

(We assume the poles are at zero)
Speer shows [Theorem 1] that the divergent expressions lie in the filtered algebra $\mathcal{M}^{\mathrm{Feyn}}(\mathbb{C}^\infty) := \cup_{k=1}^{\infty}\mathcal{M}^{\mathrm{Feyn}}(\mathbb{C}^k)$ consisting of Feynman functions $f : \mathbb{C}^k \to \mathbb{C}$,

$$f = \frac{h(z_1, \cdots, z_k)}{{L_1}^{s_1} \cdots {L_m}^{s_m}}, \quad L_i = \sum_{j \in J_i} z_j, \quad J_i \subset \{1, \cdots, k\}, \ h \text{ holom. at zero.}$$

Questions:

1. How to evaluate f consistently at the poles $z_1 = \cdots = z_k = 0$?

Brain teaser

(We assume the poles are at zero)
Speer shows [Theorem 1] that the divergent expressions lie in the filtered algebra $\mathcal{M}^{\mathrm{Feyn}}(\mathbb{C}^\infty) := \cup_{k=1}^{\infty} \mathcal{M}^{\mathrm{Feyn}}(\mathbb{C}^k)$ consisting of Feynman functions $f : \mathbb{C}^k \to \mathbb{C}$,

$$f = \frac{h(z_1, \cdots, z_k)}{{L_1}^{s_1} \cdots {L_m}^{s_m}}, \quad L_i = \sum_{j \in J_i} z_j, \quad J_i \subset \{1, \cdots, k\}, \ h \text{ holom. at zero.}$$

Questions:

1. How to evaluate f consistently at the poles $z_1 = \cdots = z_k = 0$?
2. What freedom of choice do we have for the evaluator?

Brain teaser

(We assume the poles are at zero)
Speer shows [Theorem 1] that the divergent expressions lie in the filtered algebra $\mathcal{M}^{\mathrm{Feyn}}(\mathbb{C}^\infty) := \cup_{k=1}^{\infty} \mathcal{M}^{\mathrm{Feyn}}(\mathbb{C}^k)$ consisting of Feynman functions $f : \mathbb{C}^k \to \mathbb{C}$,

$$f = \frac{h(z_1, \cdots, z_k)}{L_1{}^{s_1} \dots L_m{}^{s_m}}, \quad L_i = \sum_{j \in J_i} z_j, \quad J_i \subset \{1, \cdots, k\}, \ h \text{ holom. at zero.}$$

Questions:

1. How to evaluate f consistently at the poles $z_1 = \cdots = z_k = 0$?
2. What freedom of choice do we have for the evaluator?

Evaluating a fraction with a linear pole at zero

$$f(z_1, z_2) = \frac{z_1 - z_2}{z_1 + z_2}|_{z_1=0, z_2=0} = \begin{cases} 1 \text{ or } -1\,? \\ 0\,? \\ 10000\,? \end{cases}$$

V. Locality on meromorphic germs comes to the rescue

Locality on multiparameter meromorphic germs

Multiparameter meromorphic germs with linear poles

- $\mathcal{M}(\mathbb{C}^k) \ni f = \frac{h(\ell_1, \cdots, \ell_n)}{L_1^{s_1} \cdots L_n^{s_n}}$, h holomorphic germ, $s_i \in \mathbb{Z}_{\geq 0}$,

Locality on multiparameter meromorphic germs

Multiparameter meromorphic germs with linear poles

- $\mathcal{M}(\mathbb{C}^k) \ni f = \frac{h(\ell_1, \cdots, \ell_n)}{L_1^{s_1} \cdots L_n^{s_n}}$, h holomorphic germ, $s_i \in \mathbb{Z}_{\geq 0}$,
- $\ell_i : \mathbb{C}^k \to \mathbb{C}$, $L_j : \mathbb{C}^k \to \mathbb{C}$ linear forms.

Locality on multiparameter meromorphic germs

Multiparameter meromorphic germs with linear poles

- $\mathcal{M}(\mathbb{C}^k) \ni f = \frac{h(\ell_1,\cdots,\ell_n)}{L_1^{s_1}\cdots L_n^{s_n}}$, h holomorphic germ, $s_i \in \mathbb{Z}_{\geq 0}$,
- $\ell_i : \mathbb{C}^k \to \mathbb{C}$, $L_j : \mathbb{C}^k \to \mathbb{C}$ linear forms.
- Dependence set $\mathrm{Dep}(f) := \langle \ell_1, \cdots, \ell_m, L_1, \cdots, L_n \rangle$.

Locality on multiparameter meromorphic germs

Multiparameter meromorphic germs with linear poles

- $\mathcal{M}(\mathbb{C}^k) \ni f = \frac{h(\ell_1,\cdots,\ell_n)}{L_1^{s_1}\cdots L_n^{s_n}}$, h holomorphic germ, $s_i \in \mathbb{Z}_{\geq 0}$,
- $\ell_i : \mathbb{C}^k \to \mathbb{C}$, $L_j : \mathbb{C}^k \to \mathbb{C}$ linear forms.
- Dependence set $\mathrm{Dep}(f) := \langle \ell_1, \cdots, \ell_m, L_1, \cdots, L_n \rangle$.

Locality: separation of variables

On $\mathcal{M}(\mathbb{C}^\infty) = \bigcup_{k\in\mathbb{N}} \mathcal{M}(\mathbb{C}^k)$, $f_1 \perp f_2 \Longleftrightarrow \mathrm{Dep}(f_1) \perp \mathrm{Dep}(f_2)$.

Locality on multiparameter meromorphic germs

Multiparameter meromorphic germs with linear poles

- $\mathcal{M}(\mathbb{C}^k) \ni f = \frac{h(\ell_1, \cdots, \ell_n)}{L_1^{s_1} \cdots L_n^{s_n}}$, h holomorphic germ, $s_i \in \mathbb{Z}_{\geq 0}$,
- $\ell_i : \mathbb{C}^k \to \mathbb{C}$, $L_j : \mathbb{C}^k \to \mathbb{C}$ linear forms.
- Dependence set $\mathrm{Dep}(f) := \langle \ell_1, \cdots, \ell_m, L_1, \cdots, L_n \rangle$.

Locality: separation of variables

On $\mathcal{M}(\mathbb{C}^\infty) = \bigcup_{k \in \mathbb{N}} \mathcal{M}(\mathbb{C}^k)$, $f_1 \perp f_2 \Longleftrightarrow \mathrm{Dep}(f_1) \perp \mathrm{Dep}(f_2)$.

$\mathcal{M}_-{}^Q(\mathbb{C}^k)$ is the set of polar germs $f = \frac{h}{g}$ with $h \perp^Q g$.

Locality on multiparameter meromorphic germs

Multiparameter meromorphic germs with linear poles

- $\mathcal{M}(\mathbb{C}^k) \ni f = \frac{h(\ell_1,\cdots,\ell_n)}{L_1^{s_1}\cdots L_n^{s_n}}$, h holomorphic germ, $s_i \in \mathbb{Z}_{\geq 0}$,
- $\ell_i : \mathbb{C}^k \to \mathbb{C}$, $L_j : \mathbb{C}^k \to \mathbb{C}$ linear forms.
- Dependence set $\mathrm{Dep}(f) := \langle \ell_1, \cdots, \ell_m, L_1, \cdots, L_n \rangle$.

Locality: separation of variables

On $\mathcal{M}(\mathbb{C}^\infty) = \bigcup_{k\in\mathbb{N}} \mathcal{M}(\mathbb{C}^k)$, $f_1 \perp f_2 \Longleftrightarrow \mathrm{Dep}(f_1) \perp \mathrm{Dep}(f_2)$.

$\mathcal{M}_-{}^Q(\mathbb{C}^k)$ is the set of polar germs $f = \frac{h}{g}$ with $h \perp^Q g$.

Back to the brain teaser

$\ell := z_1 \perp z_2 =: L \Longrightarrow \frac{z_1}{z_2} \in \mathcal{M}_-^Q(\mathbb{C}^2)$

Locality on multiparameter meromorphic germs

Multiparameter meromorphic germs with linear poles

- $\mathcal{M}(\mathbb{C}^k) \ni f = \frac{h(\ell_1, \cdots, \ell_n)}{L_1^{s_1} \cdots L_n^{s_n}}$, h holomorphic germ, $s_i \in \mathbb{Z}_{\geq 0}$,
- $\ell_i : \mathbb{C}^k \to \mathbb{C}$, $L_j : \mathbb{C}^k \to \mathbb{C}$ linear forms.
- Dependence set $\mathrm{Dep}(f) := \langle \ell_1, \cdots, \ell_m, L_1, \cdots, L_n \rangle$.

Locality: separation of variables

On $\mathcal{M}(\mathbb{C}^\infty) = \bigcup_{k\in\mathbb{N}} \mathcal{M}(\mathbb{C}^k)$, $f_1 \perp f_2 \Longleftrightarrow \mathrm{Dep}(f_1) \perp \mathrm{Dep}(f_2)$.

$\mathcal{M}_-{}^Q(\mathbb{C}^k)$ is the set of polar germs $f = \frac{h}{g}$ with $h \perp^Q g$.

Back to the brain teaser

$\ell := z_1 \perp z_2 =: L \Longrightarrow \frac{z_1}{z_2} \in \mathcal{M}_-^Q(\mathbb{C}^2)$

$(\ell := z_1 - z_2) \perp (z_1 + z_2 =: L) \Longrightarrow \frac{z_1 - z_2}{z_1 + z_2} \in \mathcal{M}_-^Q(\mathbb{C}^2)$.

Back to the locality principle in QFT

We consider $\mathcal{M} := \mathcal{M}(\mathbb{C}^\infty) := \cup_{k=1}^{\infty} \mathcal{M}(\mathbb{C}^k)$ consisting of meromorphic functions/germs $f : \mathbb{C}^k \to \mathbb{C}$ with linear poles at zero,

Back to the locality principle in QFT

We consider $\mathcal{M} := \mathcal{M}(\mathbb{C}^\infty) := \cup_{k=1}^\infty \mathcal{M}(\mathbb{C}^k)$ consisting of meromorphic functions/germs $f : \mathbb{C}^k \to \mathbb{C}$ with linear poles at zero,

$$f = \frac{h(z_1, \cdots, z_k)}{{L_1}^{s_1} \cdots {L_m}^{s_m}}, \quad L_i \text{ linear in } z_1, \cdots, z_k, \quad h \text{ holom. at zero.}$$

Back to the locality principle in QFT

We consider $\mathcal{M} := \mathcal{M}(\mathbb{C}^\infty) := \cup_{k=1}^\infty \mathcal{M}(\mathbb{C}^k)$ consisting of meromorphic functions/germs $f : \mathbb{C}^k \to \mathbb{C}$ with linear poles at zero,

$$f = \frac{h(z_1, \cdots, z_k)}{{L_1}^{s_1} \cdots {L_m}^{s_m}}, \quad L_i \text{ linear in } z_1, \cdots, z_k, \quad h \text{ holom. at zero.}$$

Aim: evaluate meromorphic germs at poles according to the principle of locality: "*two events separated in space can be measured independently.*"

Back to the locality principle in QFT

We consider $\mathcal{M} := \mathcal{M}(\mathbb{C}^\infty) := \cup_{k=1}^{\infty} \mathcal{M}(\mathbb{C}^k)$ consisting of meromorphic functions/germs $f : \mathbb{C}^k \to \mathbb{C}$ with linear poles at zero,

$$f = \frac{h(z_1, \cdots, z_k)}{{L_1}^{s_1} \cdots {L_m}^{s_m}}, \quad L_i \text{ linear in } z_1, \cdots, z_k, \quad h \text{ holom. at zero.}$$

Aim: evaluate meromorphic germs at poles according to the principle of locality: "*two events separated in space can be measured independently.*"

Principle of locality: factorisation on independent events

$$\underbrace{a \text{ and } b}_{\in \mathcal{A}} \textit{ independent} \underset{\text{factorisation}}{\Longrightarrow} \mathrm{Meas} \underbrace{(a \vee b)}_{\text{concatenation}} = \mathrm{Meas}(a) \cdot \mathrm{Meas}(b).$$

Back to the locality principle in QFT

We consider $\mathcal{M} := \mathcal{M}(\mathbb{C}^\infty) := \cup_{k=1}^{\infty}\mathcal{M}(\mathbb{C}^k)$ consisting of meromorphic functions/germs $f : \mathbb{C}^k \to \mathbb{C}$ with linear poles at zero,

$$f = \frac{h(z_1, \cdots, z_k)}{{L_1}^{s_1} \cdots {L_m}^{s_m}}, \quad L_i \text{ linear in } z_1, \cdots, z_k, \quad h \text{ holom. at zero.}$$

Aim: evaluate meromorphic germs at poles according to the principle of locality: "*two events separated in space can be measured independently.*"

Principle of locality: factorisation on independent events

$$\underbrace{a \text{ and } b}_{\in\mathcal{A}} \textit{ independent} \underset{\text{factorisation}}{\Longrightarrow} \text{Meas}\underbrace{(a \vee b)}_{\text{concatenation}} = \text{Meas}(a) \cdot \text{Meas}(b).$$

- As before, we equip $\mathcal{M}$ with the **locality** relation $\perp$;

Back to the locality principle in QFT

We consider $\mathcal{M} := \mathcal{M}(\mathbb{C}^\infty) := \cup_{k=1}^{\infty}\mathcal{M}(\mathbb{C}^k)$ consisting of meromorphic functions/germs $f : \mathbb{C}^k \to \mathbb{C}$ with linear poles at zero,

$$f = \frac{h(z_1, \cdots, z_k)}{{L_1}^{s_1} \cdots {L_m}^{s_m}}, \quad L_i \text{ linear in } z_1, \cdots, z_k, \quad h \text{ holom. at zero.}$$

Aim: evaluate meromorphic germs at poles according to the principle of locality: "*two events separated in space can be measured independently.*"

Principle of locality: factorisation on independent events

$$\underbrace{a \text{ and } b}_{\in\mathcal{A}} \textit{ independent} \underset{\text{factorisation}}{\Longrightarrow} \mathrm{Meas}\ \underbrace{(a \vee b)}_{\text{concatenation}} = \mathrm{Meas}(a) \cdot \mathrm{Meas}(b).$$

- As before, we equip $\mathcal{M}$ with the **locality** relation $\perp$;

Principle of locality revisited: locality evaluators

$f \perp g \Longrightarrow \mathcal{E}(f \cdot g) = \mathcal{E}(f)\,\mathcal{E}(g)$ for two meromorphic germs f and g in an appropriate subalgebra $\mathcal{M}^\bullet$ of $\mathcal{M}$.

Speer's generalised evaluators

They consist of a family $\mathcal{E} = \{\mathcal{E}_k, \in \mathbb{N}\}$ of linear forms $\mathcal{E}_k : \mathcal{M}^{\text{Feyn}}(\mathbb{C}^k) \to \mathbb{C},$ compatible with the filtration, which fulfill the following conditions

Speer's generalised evaluators

They consist of a family $\mathcal{E} = \{\mathcal{E}_k, \in \mathbb{N}\}$ of linear forms $\mathcal{E}_k : \mathcal{M}^{\mathrm{Feyn}}(\mathbb{C}^k) \to \mathbb{C}$, compatible with the filtration, which fulfill the following conditions

1. **(extend ev_0)** $\mathcal{E}$ is the ordinary evaluation ev_0 at zero on holom. germs;

Speer's generalised evaluators

They consist of a family $\mathcal{E} = \{\mathcal{E}_k, \in \mathbb{N}\}$ of linear forms $\mathcal{E}_k : \mathcal{M}^{\mathrm{Feyn}}(\mathbb{C}^k) \to \mathbb{C},$ compatible with the filtration, which fulfill the following conditions

1. **(extend ev_0)** $\mathcal{E}$ is the ordinary evaluation ev_0 at zero on holom. germs;
2. **(partial multiplicativity)** $\mathcal{E}(f_1 \cdot f_2) = \mathcal{E}(f_1) \cdot \mathcal{E}(f_2)$ if f_1 and f_2 depend on different sets (we call them independent) of variables z_i;

Speer's generalised evaluators

They consist of a family $\mathcal{E} = \{\mathcal{E}_k, \in \mathbb{N}\}$ of linear forms $\mathcal{E}_k : \mathcal{M}^{\mathrm{Feyn}}(\mathbb{C}^k) \to \mathbb{C},$ compatible with the filtration, which fulfill the following conditions

1. **(extend ev_0)** $\mathcal{E}$ is the ordinary evaluation ev_0 at zero on holom. germs;
2. **(partial multiplicativity)** $\mathcal{E}(f_1 \cdot f_2) = \mathcal{E}(f_1) \cdot \mathcal{E}(f_2)$ if f_1 and f_2 depend on different sets (we call them independent) of variables z_i;
3. $\mathcal{E}$ is invariant under permutations of the variables $\mathcal{E}_k \circ \sigma^* = \mathcal{E}_k$ for any $\sigma \in \Sigma_k$, with $\sigma^* f(z_1, \cdots, z_k) := f(z_{\sigma(1)}, \cdots, z_{\sigma(k)})$;

Speer's generalised evaluators

They consist of a family $\mathcal{E} = \{\mathcal{E}_k, \in \mathbb{N}\}$ of linear forms $\mathcal{E}_k : \mathcal{M}^{\mathrm{Feyn}}(\mathbb{C}^k) \to \mathbb{C}$, compatible with the filtration, which fulfill the following conditions

1. **(extend ev_0)** $\mathcal{E}$ is the ordinary evaluation ev_0 at zero on holom. germs;
2. **(partial multiplicativity)** $\mathcal{E}(f_1 \cdot f_2) = \mathcal{E}(f_1) \cdot \mathcal{E}(f_2)$ if f_1 and f_2 depend on different sets (we call them independent) of variables z_i;
3. $\mathcal{E}$ is invariant under permutations of the variables $\mathcal{E}_k \circ \sigma^* = \mathcal{E}_k$ for any $\sigma \in \Sigma_k$, with $\sigma^* f(z_1, \cdots, z_k) := f(z_{\sigma(1)}, \cdots, z_{\sigma(k)})$;
4. (continuity) If $f_n(\vec{z}_k) \cdot L_1^{s_1} \cdots L_m^{s_m} \underset{n\to\infty}{\overset{\text{uniformly}}{\longrightarrow}} g(\vec{z}_k)$ as holomorphic germs, then $\mathcal{E}_k(f_n) \underset{n\to\infty}{\longrightarrow} \mathcal{E}_k(\lim_{n\to\infty} f_n)$.

Speer's generalised evaluators

They consist of a family $\mathcal{E} = \{\mathcal{E}_k, \in \mathbb{N}\}$ of linear forms $\mathcal{E}_k : \mathcal{M}^{\rm Feyn}(\mathbb{C}^k) \to \mathbb{C}$, compatible with the filtration, which fulfill the following conditions

1. **(extend ev_0)** $\mathcal{E}$ is the ordinary evaluation ev_0 at zero on holom. germs;
2. **(partial multiplicativity)** $\mathcal{E}(f_1 \cdot f_2) = \mathcal{E}(f_1) \cdot \mathcal{E}(f_2)$ if f_1 and f_2 depend on different sets (we call them independent) of variables z_i;
3. $\mathcal{E}$ is invariant under permutations of the variables $\mathcal{E}_k \circ \sigma^* = \mathcal{E}_k$ for any $\sigma \in \Sigma_k$, with $\sigma^* f(z_1, \cdots, z_k) := f(z_{\sigma(1)}, \cdots, z_{\sigma(k)})$;
4. (continuity) If $f_n(\vec{z}_k) \cdot L_1^{s_1} \cdots L_m^{s_m} \xrightarrow[n\to\infty]{\text{uniformly}} g(\vec{z}_k)$ as holomorphic germs, then $\mathcal{E}_k(f_n) \underset{n\to\infty}{\longrightarrow} \mathcal{E}_k(\lim_{n\to\infty} f_n)$.

Drawback: Speer's approach depends on the choice of coordinates $z_1, \cdots, z_k, \cdots$.

Where we stand

Data

- $(\mathcal{M}^\bullet, \perp^Q)$ an (locality) algebra of meromorphic germs at zero with a prescribed type of poles (e.g. Chen $\subset$ Speer $\subset$ Feynman);

Where we stand

Data

- $(\mathcal{M}^\bullet, \perp^Q)$ an (locality) algebra of meromorphic germs at zero with a prescribed type of poles (e.g. Chen $\subset$ Speer $\subset$ Feynman);
- $\mathcal{M}_+ \subset \mathcal{M}^\bullet$ the algebra of holomorphic germs at zero;
- the evaluation at zero: $\mathrm{ev}_0 : \mathcal{M}_+ \to \mathbb{C}$;

Where we stand

Data

- $(\mathcal{M}^\bullet, \perp^Q)$ an (locality) algebra of meromorphic germs at zero with a prescribed type of poles (e.g. Chen $\subset$ Speer $\subset$ Feynman);
- $\mathcal{M}_+ \subset \mathcal{M}^\bullet$ the algebra of holomorphic germs at zero;
- the evaluation at zero: $\mathrm{ev}_0 : \mathcal{M}_+ \to \mathbb{C}$;
- the Galois group $\mathrm{Gal}^Q(\mathcal{M}^\bullet/\mathcal{M}_+)$ of (locality) isomorphisms of $(\mathcal{M}^\bullet, \perp^Q)$;

Where we stand

Data

- $(\mathcal{M}^\bullet, \perp^Q)$ an (locality) algebra of meromorphic germs at zero with a prescribed type of poles (e.g. Chen $\subset$ Speer $\subset$ Feynman);
- $\mathcal{M}_+ \subset \mathcal{M}^\bullet$ the algebra of holomorphic germs at zero;
- the evaluation at zero: $\mathrm{ev}_0 : \mathcal{M}_+ \to \mathbb{C}$;
- the Galois group $\mathrm{Gal}^Q(\mathcal{M}^\bullet/\mathcal{M}_+)$ of (locality) isomorphisms of $(\mathcal{M}^\bullet, \perp^Q)$;
- $\mathcal{M}_-^{\bullet Q}$ is generated by polar germs $f = \frac{h}{g}$ with $h \perp^Q g$.

Where we stand

Data

- $(\mathcal{M}^\bullet, \perp^Q)$ an (locality) algebra of meromorphic germs at zero with a prescribed type of poles (e.g. Chen $\subset$ Speer $\subset$ Feynman);
- $\mathcal{M}_+ \subset \mathcal{M}^\bullet$ the algebra of holomorphic germs at zero;
- the evaluation at zero: $\mathrm{ev}_0 : \mathcal{M}_+ \to \mathbb{C}$;
- the Galois group $\mathrm{Gal}^Q(\mathcal{M}^\bullet/\mathcal{M}_+)$ of (locality) isomorphisms of $(\mathcal{M}^\bullet, \perp^Q)$;
- $\mathcal{M}_-^{\bullet Q}$ is generated by polar germs $f = \frac{h}{g}$ with $h \perp^Q g$.

Orthogonal projection

$\perp^Q$ induces a splitting

$$\mathcal{M}^\bullet = \mathcal{M}_+ \oplus^Q \mathcal{M}_-^{\bullet Q} \quad \text{and} \quad \pi_+{}^Q : \mathcal{M}^\bullet \longrightarrow \mathcal{M}_+$$

Theorem [Guo, S.P., Zhang 2022]

Definition

A locality evaluator at zero $\mathcal{E} : \mathcal{M}^\bullet \longrightarrow \mathbb{C}$ *is a linear form, which*

Theorem [Guo, S.P., Zhang 2022]

Definition

A locality evaluator at zero $\mathcal{E} : \mathcal{M}^\bullet \longrightarrow \mathbb{C}$ *is a linear form, which i) extends the ordinary evaluation* ev_0 *at zero,*

Theorem [Guo, S.P., Zhang 2022]

Definition

A locality evaluator at zero $\mathcal{E} : \mathcal{M}^\bullet \longrightarrow \mathbb{C}$ *is a linear form, which i) extends the ordinary evaluation* ev_0 *at zero, and ii) factorises on independent germs (or is a locality character):*

Theorem [Guo, S.P., Zhang 2022]

Definition

A locality evaluator at zero $\mathcal{E} : \mathcal{M}^\bullet \longrightarrow \mathbb{C}$ *is a linear form, which i) extends the ordinary evaluation* ev_0 *at zero, and ii) factorises on independent germs (or is a locality character):*

$$f_1 \perp^Q f_2 \Longrightarrow \mathcal{E}(f_1 \cdot, f_2) = \mathcal{E}(f_1) \cdot \mathcal{E}(f_2).$$

Theorem [Guo, S.P., Zhang 2022]

Definition

A locality evaluator at zero $\mathcal{E} : \mathcal{M}^\bullet \longrightarrow \mathbb{C}$ *is a linear form, which i) extends the ordinary evaluation* ev_0 *at zero, and ii) factorises on independent germs (or is a locality character):*

$$f_1 \perp^Q f_2 \Longrightarrow \mathcal{E}(f_1 \cdot, f_2) = \mathcal{E}(f_1) \cdot \mathcal{E}(f_2).$$

Example: Minimal subtraction scheme:

$$\mathcal{E}^{\mathrm{MS}} : \quad \mathcal{M}^\bullet \xrightarrow{\pi_+^Q} \mathcal{M}_+ \xrightarrow{\mathrm{ev}_0} \mathbb{C} \text{ is a locality evaluator.}$$

Theorem [Guo, S.P., Zhang 2022]

Definition

A locality evaluator at zero $\mathcal{E} : \mathcal{M}^\bullet \longrightarrow \mathbb{C}$ *is a linear form, which i) extends the ordinary evaluation* ev_0 *at zero, and ii) factorises on independent germs (or is a locality character):*

$$f_1 \perp^Q f_2 \Longrightarrow \mathcal{E}(f_1 \cdot, f_2) = \mathcal{E}(f_1) \cdot \mathcal{E}(f_2).$$

Example: Minimal subtraction scheme:

$$\mathcal{E}^{\mathrm{MS}} : \quad \mathcal{M}^\bullet \xrightarrow{\pi_+^Q} \mathcal{M}_+ \xrightarrow{\mathrm{ev}_0} \mathbb{C}$$ is a locality evaluator.

Theorem

A locality evaluator at zero $\mathcal{E} : \mathcal{M}^\bullet \longrightarrow \mathbb{C}$ *is of the form:*

Theorem [Guo, S.P., Zhang 2022]

Definition

A locality evaluator at zero $\mathcal{E} : \mathcal{M}^\bullet \longrightarrow \mathbb{C}$ *is a linear form, which i) extends the ordinary evaluation* ev_0 *at zero, and ii) factorises on independent germs (or is a locality character):*

$$f_1 \perp^Q f_2 \Longrightarrow \mathcal{E}(f_1 \cdot, f_2) = \mathcal{E}(f_1) \cdot \mathcal{E}(f_2).$$

Example: Minimal subtraction scheme:

$$\mathcal{E}^{\mathrm{MS}} : \quad \mathcal{M}^\bullet \xrightarrow{\pi_+{}^Q} \mathcal{M}_+ \xrightarrow{\mathrm{ev}_0} \mathbb{C} \text{ is a locality evaluator.}$$

Theorem

A locality evaluator at zero $\mathcal{E} : \mathcal{M}^\bullet \longrightarrow \mathbb{C}$ *is of the form:*

$$\mathcal{E} = \underbrace{\mathrm{ev}_0 \circ \pi_+{}^Q}_{\mathcal{E}^{\mathrm{MS}}} \quad \circ \quad \underbrace{T_{\mathcal{E}}}_{\mathrm{Gal}^Q(\mathcal{M}^\bullet/\mathcal{M}_+)} \quad .$$

THANK YOU FOR YOUR ATTENTION!

- P. Clavier, L. Guo, B. Zhang and S. P., An algebraic formulation of the locality principle in renormalisation, *European Journal of Mathematics*, Volume 5 (2019) 356-394

- P. Clavier, L. Guo, B. Zhang and S. P., Renormalisation via locality morphisms, *Revista Colombiana de Matemáticas*, Volume 53 (2019) 113-141

- P. Clavier, L. Guo, B. Zhang and S. P., Renormalisation and locality: branched zeta values, in "Algebraic Combinatorics, Resurgence, Moulds and Applications (Carma)" Vol. 2 ,Eds. F. Chapoton, F. Fauvet, C. Malvenuto, J.-Y. Thibon, Irma Lectures in Mathematics and Theoretical Physics **32**, *European Math. Soc.* (2020) 85–132

- P. Clavier, L. Guo, B. Zhang and S. P., Locality and renormalisation: universal properties and integrals on trees, *Journal of Mathematical Physics* **61**, 022301 (2020)

- L. Guo, B. Zhang and S. P., Renormalisation and the Euler-Maclaurin formula on cones, *Duke Math J.*, **166** (3) (2017) 537–571.
- L. Guo, B. Zhang and S. P., A conical approach to Laurent expansions for multivariate meromorphic germs with linear poles, *Pacific Journal of Mathematics* 307 (2020) 159–196.
- L. Guo, B. Zhang and S. P., Galois groups of meromorphic germs and multiparameter renormalisation (2022) (Preprint)
- L. Guo, B. Zhang and S. P., Mathematical reflections on locality (2022) (Preprint)
- R. Dahmen, A. Schmeding and S. P., A topological splitting of the space of meromorphic germs in several variables and continuous evaluators, arXiv:2206.13993 (2022)