Scalar curvature rigidity of Einstein manifolds

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2022-09-19

Global Analysis on Manifolds, Baerfest Freiburg 2022

Joint work with Klaus Kröncke
https://arxiv.org/abs/2109.09556

Motivation: Witten's proof of the PMT and rigidity

- (M, g) asymptotically Euclidean, spin, $\mathrm{scal}^g \geq 0$.
- Dirac operator D on spinors, Schrödinger-Lichnerowicz formula

$$D^2 = \nabla^* \nabla + \frac{1}{4} \mathrm{scal}\,.$$

- Euclidean metric δ on $\mathbb{R}^n$ has parallel spinors ϕ_0.
- Solve for ϕ with $D^g \phi = 0$, $\phi - \phi_0 \to 0$ at ∞.
- Integrated S-L:

$$\int_M \left(|\nabla \phi|^2 + \frac{1}{4} \mathrm{scal}\, |\phi|^2 - |D\phi|^2 \right) dV = \int_{S_\infty} Q(\phi) dS = \text{mass of } (M, g)$$

- $\Rightarrow$ mass ≥ 0 and $= 0$ only if ϕ is parallel, that is if g is Euclidean.
- In particular, $g = \delta$ outside compact $\Rightarrow g = \delta$ everywhere.

Scalar curvature rigidity

- $\Rightarrow$ $(\mathbb{R}^n, \delta)$ is *scalar curvature rigid*: it is not possible to change the metric $\delta \rightsquigarrow g$ on a compact set so that $\mathrm{scal}^g \geq \mathrm{scal}^\delta$ everywhere and $\mathrm{scal}^g > \mathrm{scal}^\delta$ somewhere.
- The "Geroch conjecture" for $(\mathbb{R}^n, \delta)$.
- *Which manifolds are SCR?*
- First variation of scalar curvature

$$D_g \,\mathrm{scal}(h) = \frac{d}{dt} \mathrm{scal}^{g+th} \,|_{t=0} = \Delta \,\mathrm{tr}\, h + \delta(\delta h) - \langle \mathrm{Ric}, h \rangle.$$

- $\Rightarrow$ *Which Einstein/Ricci-flat manifolds are SCR?*
- $\Rightarrow$ *Which complete, non-compact, Ricci-flat manifolds are SCR?*

Gravitational instantons

- *Gravitational instanton*: complete, non-compact, Ricci-flat manifold
- Many examples in dim $= 4$.
  Asymptotic geometries AE,ALE,AF,ALF,ALG,ALH $\sim$ volume growth at ∞.
- Hyperkähler examples, with parallel spinors:
  - Taub-NUT
  - Gibbons-Hawking
  - Eguchi–Hanson
  - ...

  Complete classification: Kronheimer, Minerbe, Chen-Chen.
- Generalize Witten's argument:
  SCR for instantons with parallel spinors, D.: ALE, Hein: Taub-NUT

- Examples with no parallel spinors:
  - Riemannian Schwarzschild (/Kerr)

$$g^{RS} = \left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 \sigma,$$

on $S^1(8\pi m) \times (2m, \infty) \times S^2 \rightsquigarrow S^2 \times \mathbb{R}^2$.

  - Taub-Bolt

$$g^{TB} = \frac{r^2 - m^2}{r^2 - \frac{5}{2}mr + m^2} dr^2 + (r^2 - m^2)(\sigma_1^2 + \sigma_2^2) + 4m^2 \frac{r^2 - \frac{5}{2}mr + m^2}{r^2 - m^2} \sigma_3^2,$$

on $(2m, \infty) \times S^3 \rightsquigarrow \mathbb{C}P^2 \setminus \{\text{point}\}$.

  - Chen-Teo instanton.

All with special geometry: One-sided type D/Hermitian. Classification??

Stability of Einstein manifolds

- Einstein metrics are critical points of Hilbert functional

$$g \mapsto \int \mathrm{scal}^g \, dV^g$$

  w.r.t. volume-preserving variations.
- TT-tensors: trace- and divergence-free 2-tensors.
- Einstein operator on 2-tensors: $\Delta_E = \nabla^*\nabla - 2\mathring{R}$, where $\mathring{R}h_{ij} = h^{kl}R_{iklj}$
- (Open) Einstein manifold is *stable* if

$$\mu_1(\Delta_E|_{TT}, M) = \inf\left\{(\Delta_E h, h)_{L^2} \mid h \in C_c^\infty(TT), \|h\|_{L^2}^2 = 1\right\} \geq 0$$

  and *unstable* otherwise.
- Thm. (Wang, Dai-Wang-Wei): If (M, g) (compact) has parallel spinors, then it is stable. $\rightsquigarrow \Delta_E \sim (D^{T^*M})^2 \geq 0$.
- The Riemannian Schwarzschild and the Taub-Bolt metrics are unstable.

Compactly supported perturbations

Thm 1 (D.-Kröncke)

Assume (M, g) compact Einstein with boundary, not locally a warped product. Let f_t be a 1-parameter family of smooth functions with supports in relatively compact Ω. Assume $h \neq 0$ TT-tensor with support in Ω satisfying

$$\int_M \langle \Delta_E h, h \rangle \, dV = -2 \int_M f_0 \, dV. \tag{1}$$

Then there is a family g_t of metrics with $g_0 = g$ and $\frac{d}{dt} g_t|_{t=0} = h$ such that

$$\mathrm{scal}^{g_t} = \mathrm{scal}^g + \frac{t^2}{2} f_t, \tag{2}$$

$$dV^{g_t} = dV^g, \tag{3}$$

and $g_t = g$ outside of Ω.

- Analogue to Kazdan-Warner trichotomy. For compact manifolds:
  - (i) Any smooth function is the scalar curvature of a smooth metric.
  - (ii) A smooth function is the scalar curvature of a smooth metric if it is either identically zero or strictly negative somewhere. In this case, any scalar-flat metric is Ricci-flat.
  - (iii) A smooth function is the scalar curvature of a smooth metric if it is strictly negative somewhere.

  Dichotomy for perturbations of Einstein metrics:
  - (i)' If (M, g) is linearly unstable, any C_c^∞-function is a second order scalar curvature perturbation of a $C_c^\infty(TT)$-perturbation of g.
  - (iii)' If (M, g) is linearly stable, a C_c^∞-function is a second order scalar curvature perturbation of a $C_c^\infty(TT)$-perturbation of g if and only if it has negative integral.

  No analogue of (ii): In the *neutrally lineary stable* case $\mu_1^D(\Delta_E, TT) = 0$, the infimum in $\mu_1^D(\Delta_E, TT)$ not realized by a C_c^∞-tensor.

Proof: background

- Variation of scalar curvature, $\mathrm{Ric} = \sigma g$,

$$D_g \operatorname{scal}(h) = \frac{d}{dt} \operatorname{scal}^{g+th} |_{t=0} = \Delta \operatorname{tr} h + \delta(\delta h) - \sigma \operatorname{tr} h.$$

and if h is TT,

$$\begin{aligned} D_g^2 \operatorname{scal}(h,h) &= \frac{d^2}{dt^2} \operatorname{scal}^{g+th} |_{t=0} \\ &= -\Delta(|h|^2) + \delta(\delta' h) - \frac{1}{2}\langle \Delta_E h, h\rangle + \sigma |h|^2. \end{aligned}$$

- Second divergence $P = \delta^2$ is an *underdetermined* elliptic operator.
  $\rightsquigarrow$ surjective but not injective principal symbol

- Second divergence $P = \delta^2$ is an *underdetermined* elliptic operator from trace-free symmetric 2-tensors to functions.
- $P^* = \mathring{\nabla}^2$ trace-free Hessian.

Thm (Delay)

For (M, g), assume kernel of P^* are only constants.
Then for any compactly supported f with $\int_\Omega f \; dV = 0$, there is a compactly supported trace-free two-tensor U solving

$$\delta(\delta U) = f.$$

- Assumption is equivalent to (M, g) not a warped product.

Proof: constructing compactly supported perturbations

- Proof step 1: Construct solution to 2nd order.
- Set $g_t := g + th + \frac{t^2}{2}k$ with given h, find k!
- Since h is TT:

$$\frac{d}{dt}\operatorname{scal}^{g_t}|_{t=0} = D_g \operatorname{scal}(h) = 0$$

$$\frac{d}{dt}dV^{g_t}|_{t=0} = D_g dV(h) = \left(\frac{1}{2}\operatorname{tr}_g h\right) dV^g = 0$$

- Solve for k such that

$$\frac{d^2}{dt^2}\operatorname{scal}^{g_t}|_{t=0} = D_g \operatorname{scal}(k) + D_g^2 \operatorname{scal}(h,h) = f_0,$$

$$\frac{d^2}{dt^2}dV^{g_t}|_{t=0} = D_g dV(k) + D_g^2 dV(h,h) = 0,$$

- trace-free part of k: scal equation, trace of k: dV equation...

- Set $k = \frac{1}{n}|h|^2 g + \mathring{k}$ where $\operatorname{tr} \mathring{k} = 0$. Then

$$D_g dV(k) + D_g^2 dV(h,h) = \left(\frac{1}{2} \operatorname{tr}^g k + \frac{1}{4} (\operatorname{tr}^g h)^2 - \frac{1}{2} |h|_g^2 \right) dV^g = 0,$$

- For scalar curvature equation we have

$$D_g \operatorname{scal}(k) + D_g^2 \operatorname{scal}(h,h) = \delta \left(\delta' h - \frac{1}{n} d|h|^2 + \delta \mathring{k} \right) - \frac{1}{2} \langle \Delta_E h, h \rangle = f_0.$$

or

$$\delta^2 \mathring{k} = -\delta \left(\delta' h - \frac{1}{n} d|h|^2 \right) + \left(\frac{1}{2} \langle \Delta_E h, h \rangle + f_0 \right).$$

- Both terms integrate to zero, Delay's theorem gives solution $\mathring{k}$...
- $\Rightarrow$

$$\operatorname{scal}^{g_t} = \operatorname{scal}^g + \frac{t^2}{2} f_t + O(t^3)$$

$$dV^{g_t} = dV^g + O(t^3)$$

- Proof step 2: Iteration to exact solution. Compare implicit function theorem.
- Set

$$g_t^{(i)} := g + h_t^{(i)} + \frac{1}{2} k_t^{(i)}$$

with $h_t^{(0)} := th$ and $k_t^{(0)} := t^2 k$.

- Recursive formulas:

$$\begin{aligned} D_g \operatorname{scal}(k_t^{(i+1)} - k_t^{(i)}) + D_g^2 \operatorname{scal}(h_t^{(i+1)}, h_t^{(i+1)}) - D_g^2 \operatorname{scal}(h_t^{(i)}, h_t^{(i)}) \\ = \operatorname{scal}(g) + \frac{t^2}{2} f_t - \operatorname{scal}(g_t^{(i)}) \end{aligned}$$

and

$$\begin{aligned} D_g dV(k_t^{(i+1)} - k_t^{(i)}) + D_g^2 dV(h_t^{(i+1)}, h_t^{(i+1)}) - D_g^2 dV(h_t^{(i)}, h_t^{(i)}) \\ = dV(g) - dV(g_t^{(i)}) \end{aligned}$$

- Meaning of recursive formulas?
- Set

$$g_{t,s}^{(i)} := g + sh_t^{(i)} + \frac{s^2}{2} k_t^{(i)}.$$

Then

$$\frac{d}{ds} \operatorname{scal}(g_{t,s}^{(i)})|_{s=0} = 0,$$

$$\frac{d^2}{ds^2} \operatorname{scal}(g_{t,s}^{(i)})|_{s=0} = D_g \operatorname{scal}(k_t^{(i)}) + D_g^2 \operatorname{scal}(h_t^{(i)}, h_t^{(i)}).$$

- Taylor expansion in s:

$$\begin{aligned} \operatorname{scal}(g_t^{(i)}) &= \operatorname{scal}(g_{t,1}^{(i)}) \\ &= \operatorname{scal}(g) + D_g \operatorname{scal}(k_t^{(i)}) + D_g^2 \operatorname{scal}(h_t^{(i)}, h_t^{(i)}) + S_t^{(i)} \end{aligned}$$

where $S_t^{(i)}$ is Taylor remainder term.

- scal recursion:

$$D_g \operatorname{scal}(k_t^{(i+1)}) + D_g^2 \operatorname{scal}(h_t^{(i+1)}, h_t^{(i+1)}) = \frac{t^2}{2} f_t - S_t^{(i)},$$

or

$$\operatorname{scal}(g_t^{(i+1)}) = \operatorname{scal}(g) + \frac{t^2}{2} f_t + S_t^{(i+1)} - S_t^{(i)},$$

- Similar for dV

$$\begin{aligned} dV(g_t^{(i)}) &= dV(g_{t,1}^{(i)}) \\ &= dV(g) + D_g dV(k_t^{(i)}) + D_g^2 dV(h_t^{(i)}, h_t^{(i)}) + V_t^{(i)} \end{aligned}$$

where $V_t^{(i)}$ is Taylor remainder term.

- dV recursion:

$$D_g dV(k_t^{(i+1)}) + D_g^2 dV(h_t^{(i+1)}, h_t^{(i+1)}) = -V_t^{(i)}$$

or

$$dV(g_t^{(i+1)}) = dV(g) + V_t^{(i+1)} - V_t^{(i)},$$

- Proof step 3: Solve for $h_t^{(i)}$, $k_t^{(i)}$ just as in step 1.
- Proof step 4: Prove contraction estimates

$$\|(h_t^{(i+2)} - h_t^{(i+1)})\| + \|k_t^{(i+2)} - k_t^{(i+1)}\| \\ \leq \frac{1}{3}\left(\|(h_t^{(i+1)} - h_t^{(i)})\| + \|k_t^{(i+1)} - k_t^{(i)}\|\right).$$

  for small $t \Rightarrow$ convergence.
- Proof step 5: Prove regularity of limit.

SCR and stability

- We have proven one half of:

Thm 2 (D.-Kröncke)

Let $(M, \hat{g})$ be an open Einstein manifold which is not locally isometric to a warped product. Then the following are equivalent:

(i) $(M, \hat{g})$ is linearly stable.

(ii) Close to $\hat{g}$, there is no metric g with

$$g - \hat{g}|_{M \setminus K} \equiv 0, \qquad \mathrm{vol}(K, g) = \mathrm{vol}(K, \hat{g})$$

for some compact set $K \subset M$ which additonally satisfies

$$\mathrm{scal}^g \geq \mathrm{scal}^{\hat{g}} \qquad \mathrm{scal}^g(p) > \mathrm{scal}^{\hat{g}}(p) \text{ for some } p \in M.$$

- + Neumann eigenvalue condition for $\Delta^{\hat{g}}$ in case $\mathrm{scal}^{\hat{g}} > 0$.
- also equivalent: If g is close to $\hat{g}$ with $\mathrm{scal}^g \equiv \mathrm{scal}^{\hat{g}}$ and

$$g - \hat{g}|_{M \setminus K} \equiv 0, \qquad \mathrm{vol}(K, g) = \mathrm{vol}(K, \hat{g}),$$

then g is isometric to $\hat{g}$.

Proof: a generalized λ-functional

- Prove opposite implication in Theorem. Not a deformation argument!
- For $(M, \hat{g})$ and $\alpha > 0$, define

$$F_\alpha : \mathcal{M}^\infty_{\hat{g}} \times C^\infty(M) \to \mathbb{R}, \qquad F_\alpha(g, f) := \int_M \left(\mathrm{scal} + \alpha |\nabla f|^2\right) e^{-f} \, dV,$$

and

$$\lambda_\alpha(g) := \inf \left\{ F_\alpha(g, f) \mid f \in C^\infty(M), \int_M e^{-f} \, dV = 1 \right\}.$$

- $\alpha = 1 \Rightarrow$ Perelman λ-functional.
- α needed to handle boundary with auxiliary function satisfying a Laplace equation.
- substitution $\omega = e^{-\frac{f}{2}}$ gives

$$\lambda_\alpha(g) = \inf \left\{ G_\alpha(g, \omega) \mid \omega \in C^\infty(M), \int_M \omega^2 \, dV = 1 \right\}$$

where

$$G(g, \omega) := \int_M (4\alpha |\nabla \omega|^2 + \mathrm{scal}\, \omega^2) \, dV.$$

- scal $\geq c$ implies $\lambda_\alpha(g) \geq c$, and if scal $\not\equiv c$, then $\lambda_\alpha(g) > c$.

- First variation formula

$$D_g\lambda_\alpha(h) = -\int_M \langle \mathrm{Ric} + \nabla^2 f - (\alpha-1)\nabla f \otimes \nabla f, h\rangle e^{-f}\, dV$$
$$+ \int_M \langle \frac{1}{2}(1-\frac{1}{\alpha})(\mathrm{scal} - \lambda_\alpha(g))g + \frac{1}{2}(\alpha-1)|\nabla f|^2 g, h\rangle e^{-f}\, dV.$$

- First variation formula for g Einstein, $\mathrm{Ric} = \sigma g$,

$$D_g\lambda_\alpha(h) = -\frac{\sigma}{\mathrm{vol}(M,g)}\int_M \mathrm{tr}\, h\, dV$$

- Einstein metrics are critical w.r.t. volume-preserving deformations, Ricci-flat metrics are critical for general deformations.

- Second variation formula (volume preserving h)

$$D_g^2\lambda_\alpha(h,h) = -\frac{1}{\mathrm{vol}(M,g)}\int_M \langle \frac{1}{2}\Delta_E h - \delta^*(\delta h) - \frac{1}{2}\delta(\delta h)g, h\rangle \, dV$$
$$- \frac{1}{\mathrm{vol}(M,g)}\int_M v\delta(\delta h)\, dV$$
$$+ \frac{1}{\mathrm{vol}(M,g)}\int_M ((\alpha-1)\Delta v + \sigma v)\,\mathrm{tr}\, h\, dV,$$

where v solves

$$2\alpha\Delta v = \Delta(\mathrm{tr}\, h) + \delta(\delta h) - \sigma\,\mathrm{tr}\, h, \qquad \nabla_\nu v = 0.$$

- Second variation formula for g Einstein, $\mathrm{Ric} = \sigma g$, $h \in C_c^\infty(TT)$,

$$D_g^2\lambda_\alpha(h,h) = -\frac{1}{2\,\mathrm{vol}(g)}\int_M \langle \Delta_E h, h\rangle \, dV.$$

- $\Rightarrow$ g should be local maximum of λ_α if g is stable.

- $\mathcal{M}^\infty_{\hat{g}}$ be the set of metrics on M such that $g - \hat{g}$ vanishes to every order at ∂M
- Study $D^2_{\hat{g}}\lambda_\alpha(h,h)$ on

$$C^\infty_{c,0}(M)\hat{g} \oplus \delta^*(C^\infty_c(T^*M)) \oplus C^\infty_c(TT)$$

  (volume preserving h) for $\hat{g}$ stable Einstein. Exponentiate:
- Thm: If $(M,\hat{g})$ is stable Einstein, then there is a neighbourhood $\mathcal{V}$ of $\hat{g}$ in $\mathcal{M}^\infty_{\hat{g}}$ such that

$$\lambda_\alpha(g) \leq \lambda_\alpha(\hat{g})$$

  for every $g \in \mathcal{V}$ with $\mathrm{vol}(M,g) = \mathrm{vol}(M,\hat{g})$. Further, equality holds if and only if g is Einstein.
- Thm 2 follows.

Application

- $\Rightarrow$ The Riemannian Schwarzschild metric and the Taub-Bolt metric have compactly supported positive scalar curvature perturbations.
- conformal change to scal $= 0 \Rightarrow$ perturbation decreasing "mass" while length of circle at infinity is constant.

End

- Thank you for your attention!