Gravitational instantons and special geometry

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Baerfest, Sept. 20, 2022

joint work with Steffen Aksteiner, Mattias Dahl, Gustav Nilsson, and Walter Simon

Introduction

- Einstein equations, 4D

  Action $S = \int R d\mu$

  $\delta S = 0 \quad \Rightarrow \quad R_{ab} = 0$

- Lorentz signature $+ - - -$
  - hyperbolic (mod. gauge)
  - Cauchy problem

Major conjectures ~ 1970

- Strong/Weak C. C.

- BH Uniqueness/Stability

- P.I., End State Conjecture

Open

Introduction

Riemannian vacuum Einstein

- Riemann signature $+ + + +$:
  - elliptic (mod. gauge)
  - real analytic

Compact Ricci flat

- $K3$ surfaces are only known examples.
- Question: are those all?

Gravitational instanton

- (M, g_{ab}) complete, noncompact, $R_{ab} = 0$

  with $|\text{Riem}| = o(r^{-2}), \quad r(\cdot) = d(\cdot, p)$

  Hyperkähler is not assumed here

$\Rightarrow$ at most quartic volume growth: $\quad \text{Vol}(B_r) = O(r^\alpha), \quad \alpha \leq 4$

Introduction

Gravitational Instantons

- $\alpha = 4$:   ALE: $\partial_\infty \mathcal{M} \sim S^3/\Gamma$

- $\alpha = 3$:   ALF, AF: $\partial_\infty \mathcal{M} \sim$ circle bundle over S^2 or $\mathbb{RP}^2$

    AF: $\partial_\infty \mathcal{M} \sim$ trivial circle bundle $S^1 \times S^2$

- $\alpha \leq 2$:   ALG, ALG*, ALH, ALH*

Classical examples

- ALE: Eguchi-Hanson, Gibbons-Hawking
- ALF: (multi-)Taub-NUT, Gibbons-Hawking, Atiyah-Hitchin, Taub-bolt
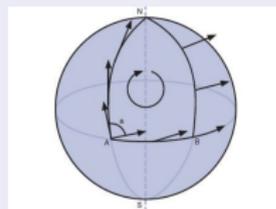- AF: Schwarzschild, Kerr

Major conjectures ~ 1980

- Euclidean BH Uniqueness: AF nonflat $\Rightarrow$ Kerr   Wrong
- ALF $\Rightarrow$ G-H, A-H, T-b   Open
- ALE $\Rightarrow$ E-H, G-H   Open

Special geometry

- $** = 1 \Rightarrow \mathcal{W} = \mathcal{W}^+ + \mathcal{W}^-, \quad \mathcal{W}^\pm$ real (anti-)self dual Weyl

Special Holonomy

- $G \subsetneq \mathrm{SO}(4)$ hyperkähler, half flat

- 2 parallel spinors, 3 parallel complex
  structures in involution

Remark

A classification of hyperkähler instantons is known Sun & Zhang (2021), Chen& Chen (2015, 2016), Minerbe (2011), Kronheimer (1989)

Special geometry

Algebraically special curvature

Algebraically special if $\#\mathrm{Spec}(\mathcal{W}^{\pm}) < 3$.

$\mathcal{W}^{\pm}$ Petrov type D if $\#\mathrm{Spec}(\mathcal{W}^{\pm}) = 2$

- $(\mathcal{M}, g_{ab})$ is *one-sided algebraically special* if one of $\mathcal{W}^{\pm}$ is algebraically special

Riemannian and complex geometry

- Hermitian: Riemannian space with integrable complex structure
- Kähler: Riemannian space with parallel complex structure

Facts:

- $R_{ab} = 0$ and one-sided algebraically special $\Rightarrow$ Hermitian
- $R_{ab} = 0$ and Hermitian $\Rightarrow$ conformally Kähler.
- $R_{ab} = 0$ and Kähler $\Rightarrow$ half flat (hyperkähler)

Examples
Euclidean Kerr Instanton

- $\Sigma = r^2 - a^2 \cos^2 \theta, \quad \Delta = r^2 - 2Mr - a^2$

$$g_{ab}dx^a dx^b = +\frac{\Delta}{\Sigma}\left(d\tau - a\sin^2\theta d\phi\right)^2 + \frac{\sin^2\theta}{\Sigma}\left(ad\tau + (r^2 - a^2)d\phi\right)^2$$
$$+ \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2,$$

- identify $(\tau, \phi) = (\tau + 2\pi/\kappa, \phi - 2\pi\Omega/\kappa)$

  with $\kappa = \frac{r_+ - r_-}{2(r_+^2 - a^2)}, \quad \Omega = \frac{a}{r_+^2 - a^2}$

  $\rightsquigarrow$ AF instanton on $\mathbb{R}^2 \times S^2 \cong S^4 \backslash S^1$.

- toric with 2 isolated fixed points, Petrov type D

Examples

Chen-Teo Instanton Y. Chen & E. Teo, 2011, 2015

- 2-parameter family of AF instantons on $\mathbb{CP}^2 \setminus S^1$
- toric, 3 isolated fixed points
- constructed using the soliton method   Belinski & Zakharov (1978, 1979)
- Counterexample to the classical Euclidean Black Hole Uniqueness Conjecture!

Updated conjecture:

Euclidean Black Hole Uniqueness Conjecture

An AF gravitational instanton on $\mathbb{R}^2 \times S^2$ is in the Kerr family.

Examples

Chen-Teo instanton Y. Chen & E. Teo, 2011, 2015

Theorem (S. Aksteiner & L.A. arXiv:2112.11863)

- *Chen-Teo is Hermitian, non-Kähler.*
- $\mathcal{W}^-$ *is algebraically general*

Remark

All known examples of gravitational instantons are Hermitian.

Instanton Classification Problem

Hermitian, toric instantons

Theorem (Biquard & Gauduchon arXiv:2112.12711)

A Hermitian, toric ALF instanton belongs to the Kerr , Chen-Teo , Taub-bolt, or Taub-NUT families.

Remark

- *LeBrun (2012) classified compact Hermitian-Einstein 4-manifolds.*
- *A compact non-Kähler Hermitian-Einstein 4-manifold is toric.*

Conjecture (L.A. & S. Aksteiner arXiv:2112.11863)

An ALF Hermitian non-Kähler instanton is toric, and hence belongs to the Kerr , Chen-Teo , or Taub-bolt families.

Instanton Classification Problem

S^1 instantons

- An S^1 instanton is a gravitational instanton with an effective isometric S^1 action generated by a Killing field with bounded norm.

- Main result of this talk:

Theorem (Aksteiner, L.A., Dahl, Nilsson, Simon *(in preparation)*)

- *An AF S^1 instanton on $S^4 \backslash S^1$ is Kerr*

- *An AF S^1 instanton on $\mathbb{CP}^2 \backslash S^1$ is Hermitian*   *⊃ Chen-Teo*

- *An ALF S^1 instanton on $\mathbb{CP}^2 \backslash \{\infty\}$ is Taub-bolt*

Remark

The proof uses an identity of Israel-Robinson type, combined with the G-signature theorem.

Nuts and bolts

- $\mathcal{M}$ oriented 4-manifold with an effective S^1-action.
- Fixed point set F consists of isolated points (nuts) $P_1, \ldots, P_{n_{\text{nuts}}}$ and surfaces (bolts) $B_1, \ldots B_{n_{\text{bolts}}}$.

Surface gravities

- Nut P: surface gravity κ^i, $i = 1, 2$. Periods $2\pi/|\kappa^i|$ for exceptional orbits near P.

$$\nabla\xi|_P = \begin{pmatrix} 0 & \kappa^1 & 0 & 0 \\ -\kappa^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \kappa^2 \\ 0 & 0 & -\kappa^2 & 0 \end{pmatrix}$$

- Bolt B: Surface gravity κ. Period $2\pi/\kappa$.

$$\nabla\xi|_B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \kappa \\ 0 & 0 & -\kappa & 0 \end{pmatrix}$$

Nuts and bolts
Weights

- At a nut P

$$\frac{|\kappa^1(P)|}{|\kappa^2(P)|} = \frac{w^1(P)}{w^2(P)}$$

  with w^1, w^2 mutually prime, non-negative integers (weights).
- A nut P with orientation, or sign, $\epsilon(P)$ and weights $w^i(P)$, $i = 1, 2$ has fixed point data $\{\epsilon(P), w^1(P), w^2(P)\}$

- Generic period $2\pi G$:

$$G = \frac{w^1(P)}{|\kappa^1(P)|} = \frac{w^2(P)}{|\kappa^2(P)|} = \frac{1}{|\kappa(B)|}$$

ALF S^1 Instantons

- (M, g_{ab}) ALF S^1-instanton with S^1 action generated by KVF ξ^a with $|\xi| \leq 1$.
- Define

$$\lambda = \xi_a \xi^a \qquad\qquad \text{norm squared}$$

$$\omega_a = \epsilon_{abcd} \nabla^c \xi^d \xi^b = \nabla_a \omega, \qquad \text{twist}$$

$$\mathcal{E}^\pm = \lambda \pm \omega, \qquad\qquad \text{Ernst potential}$$

$$\mathcal{F}^\pm{}_{ab} = \nabla_a \xi_b \pm \tfrac{1}{2} \epsilon_{ab}{}^{cd} \nabla^c \xi^d, \qquad \text{sd/asd part of } d\xi$$

$$\mathcal{F}^{\pm 2} = \mathcal{F}^\pm{}_{ab} \mathcal{F}^{\pm ab},$$

We have

$$\Delta \mathcal{E}^\pm = \mathcal{F}^{\pm 2} = \lambda^{-1} \nabla_a \mathcal{E}^\pm \nabla^a \mathcal{E}^\pm \geq 0$$

ALF S^1 Instantons

Divergence identity

- Define

$$\Psi^{\pm a} = \frac{1}{2} J^{\pm a} |\mathbf{s}^{\pm}| + \frac{(1 + \mathcal{E}^{\mp})(1 - \mathcal{E}^{\pm})}{2\lambda} \nabla^a |\mathbf{s}^{\pm}|$$

where $J^{\pm a}$ is the conserved current

$$J^{\pm a} = \frac{\pm(1 \mp \mathcal{E}^+)^2 \nabla^a \mathcal{E}^- \mp (1 \pm \mathcal{E}^-)^2 \nabla^a \mathcal{E}^+}{2\lambda^2}.$$

and $\quad \mathbf{s}^{\pm 2} = \dfrac{\mathcal{F}^{\pm 2}}{(1 - \mathcal{E}^{\pm})^4} \quad$ is the Mars-Simon scalar.

- $\Psi^{\pm a}$ is regular except at zeros of λ, $\mathbf{s}^{\pm}$

ALF S^1 Instantons

Divergence identity

- Define

$$\mathcal{S}^{\pm}_{abcd} = \mathcal{W}^{\pm}_{abcd} - \frac{6\left(\mathcal{F}^{\pm}_{ab}\mathcal{F}^{\pm}_{cd} - \frac{1}{3}\mathcal{F}^{\pm 2}\mathcal{I}^{\pm}_{abcd}\right)}{1 - \mathcal{E}^{\pm}}, \quad \text{Mars-Simon tensor}$$

$$\mathcal{S}^{\pm 2} = \mathcal{S}^{\pm}_{abcd}\mathcal{S}^{\pm abcd},$$

  where $\mathcal{I}^{\pm}_{abcd} = \frac{1}{4}(g_{ac}g_{bd} - g_{ad}g_{bc} \pm \epsilon_{abcd})$ is the metric on sd and asd 2-forms.

- Let

$$V^{\pm} = \frac{\lambda\left(\mathcal{F}^{\pm 2}\mathcal{S}^{\pm 2} - (\mathcal{F}^{\pm}\mathcal{S}^{\pm})^2\right)}{4(\mathcal{F}^{\pm 2})^2},$$

  where $(\mathcal{F}^{\pm}\mathcal{S}^{\pm})^2 = \mathcal{F}^{\pm ab}\mathcal{S}^{\pm ef}_{ab}\mathcal{F}^{\pm cd}\mathcal{S}^{\pm}_{efcd}$.

ALF S^1 Instantons

Divergence identity

Lemma (Magic Identity)

$$\nabla_a \Psi^{\pm a} = \frac{(1 + \mathcal{E}^{\mp})(1 - \mathcal{E}^{\pm})}{2\lambda} V^{\pm} |\mathbf{s}^{\pm}| \geq 0 \qquad (\dagger)$$

and equality holds $\Leftrightarrow \mathcal{W}^{\pm}{}_{abcd}$ *is algebraically special.*

Remark

($\dagger$) *is an identity of Israel-Robinson type that originates in attempts to prove the Lorentzian Black Hole Uniqueness Conjecture, cf. Simon (1995). In the Lorentzian case, the terms in the corresponding identity are complex.*

ALF S^1 Instantons

Divergence identity

Integrating the Magic Identity gives

$$0 \leq \int_{\mathcal{M}} \nabla_a \Psi^{\pm a} = \text{boundary terms at nuts, bolts, } \{\mathcal{F}^{\pm^2} = 0\}, \infty$$

RHS = 0 $\Rightarrow$ algebraically special.

Boundary terms:

- nut P: $\quad \epsilon(P) \dfrac{|\kappa^1(P) \pm \kappa^2(P)|}{|\kappa^1(P)\kappa^2(P)|}$, $\qquad$ bolt B: $\dfrac{\chi[B_i]}{|\kappa(B_i)|}$

- ∞: $\quad$ For $p \in \mathcal{M}$, let $L(p)$ be the length of the orbit through p, and define

$$\ell_\infty = \limsup_{p \to \infty} L(p)$$

The boundary term at ∞ is $-4\pi\ell_\infty$.

- The singular terms at $\{\mathcal{F}^{\pm^2} = 0\}$ have a good sign.

ALF S^1 Instantons

Divergence identity

- For a nut P with fixed point data $\{\epsilon, a, b\}$, with $a \leq b$. Let

$$Z^\pm(P) \equiv Z^\pm(\epsilon, a, b) := \pm\epsilon\frac{1}{a} + \frac{1}{b}$$

- Extract a dimensionless inequality by rescaling with generic period G

Corollary (Dimensionless magic)

Let $w \geq 1$ be the maximal weight of all nuts in $\mathcal{M}$. If $\mathcal{M}$ has no nuts, we set $w = 1$. It holds that

$$-\frac{2}{w} + \sum_{i=1}^{n_{\text{bolts}}} \chi[B_i] + \sum_{i=1}^{n_{\text{nuts}}} Z^\pm(P_i) \geq 0$$

with equality only if $\mathcal{W}^\pm{}_{abcd}$ is algebraically special.

Topological invariants
Applications of the index theorem

- $\chi[\mathcal{M}] = n_{\text{nuts}} + \sum_{i=1}^{n_{\text{bolts}}} \chi[B_i],$

- $\text{sign}[\mathcal{M}] = \sum_{i=1}^{n_{\text{nuts}}} \epsilon(P_i) + \text{sgn}(e)$ where e is the Euler number of the circle bundle at infinity.

G-signature theorem Atiyah, Hirzebruch

- Generalization to ALF case $\rightsquigarrow$ G-eta invariant at ∞

- $\mathcal{M}$ ALF S^1-instanton. Then

$$\text{sign}[\mathcal{M}] = \sum_{i=1}^{n_{\text{nuts}}} \epsilon(P_i) \prod_{j=\pm} \frac{1 + z^{w^j(P_i)}}{1 - z^{w^j(P_i)}} + \frac{4z}{(1-z)^2}\left(e - \sum_{i=1}^{n_{\text{bolts}}} B_i \cdot B_i \right) + \text{sgn}(e),$$

where z is an indeterminate.

Topological invariants

Structure of the fixed point set

Remark

- *The G-signature formula makes it possible to apply techniques from complex analysis and number theory to analyze the structure of the fixed point set Li&Liu (2011), Jang (2018)*

- *This yields a classification of the possible fixed point sets in terms of nut and bolt data.*

- *Ideas developed for compact spaces generalize to ALF case.*

Cases with few nuts Jang (2018)

2 nuts:	$\{+, a, b\}, \{-, a, b\}$	$\leftrightarrow$ Kerr
3 nuts:	$\{\pm, a, b\}, \{\mp, a, a+b\}, \{\mp, b, a+b\}$	$\leftrightarrow$ Chen-Teo

Kerr characterization

Theorem

Let $(\mathcal{M}, g_{ab})$ be an AF S^1-instanton on $S^4 \backslash S^1$. Then $(\mathcal{M}, g_{ab})$ belongs to the Kerr family.

Some steps in the proof:

- $\mathcal{M} = S^4 \backslash S^1 \quad \Rightarrow \quad \chi[\mathcal{M}] = 2, \text{sign}[\mathcal{M}] = 0, e = 0$

- Index theorem $\Rightarrow \text{sign}[\mathcal{M}] = \sum_{i=1}^{n_{\text{nuts}}} \epsilon(P_i)$

  $\Rightarrow k$ positively and k negatively oriented nuts.

- $\sum_{i=1}^{n_{\text{bolts}}} \chi[B_i] = 2 - 2k$

- Dimensionless magic $\Rightarrow$

$$-\frac{2}{w} + \sum_{i=1}^{2k} Z^{\pm}(P_i) + 2 - 2k \geq 0 \qquad (\star)$$

Kerr characterization

Sample application of the *G*-signature theorem

- Highest weight: Let $w > 1$ be the highest weight of any nut. Let P_1 be a nut with data (ϵ, a, w). Then, either there is P_2 with data $(-\epsilon, a, w)$, or with data (ϵ, b, w) and in this case $w = a + b$.

- In case P_2 has data $(-\epsilon, a, w)$, $(\star)$ gives

$$-\frac{2}{w} + Z^{\pm}(+, a, w) + Z^{\pm}(-, a, w) \geq 2k - 2 - \underbrace{\sum_{i=3}^{2k} Z^{\pm}(P_i)}_{\text{equal \# nuts \& anti-nuts}} \qquad (\star\star)$$

- Elementary inequality: $Z^{\pm}(+, a, b) + Z^{\pm}(-, c, d) \leq 2$ with equality iff $a = b = c = d = 1$.

- $Z^{\pm}(+, a, w) + Z^{\pm}(-, a, w) = \dfrac{2}{w}$

$\Rightarrow$ equality in $(\star\star)$ for $\pm$

Kerr characterization

Rigidity

Further consequences of the *G*-signature theorem

- Weight balance: A weight $w > 0$ occurs an even number of times for nuts.
- Companion nuts: Let $a, w > 1$ be weights for a nut P. There is nut Q with weights b, w where
  - If $\epsilon(Q) = \epsilon(P)$, then $a \equiv -b \mod w$
  - If $\epsilon(Q) = -\epsilon(P)$, then $a \equiv b \mod w$

- Several other cases to consider — in each case get equality in $(\star\star)$ for $\pm$

  $\Rightarrow$ Petrov type D

  $\Rightarrow$ Kerr using results of Biquard & Gauduchon arXiv:2112.12711

- Similar considerations apply to Taub-bolt and Chen-Teo cases.

Concluding remarks

- The results presented here provide a step towards the classification of gravitational instantons

- The classification problem for instantons without symmetry assumptions is challenging, cf. "Besse conjecture".

Open problems

- Show that Hermitian instantons are toric, and hence that AF S^1 instantons on $\mathbb{CP}^2 \backslash S^1$ are Chen-Teo.
- Show that a toric AF instanton is Kerr or Chen-Teo.
- Investigate rigidity of Kerr, Taub-bolt and Chen-Teo:
  Are there instantons on $S^4 \backslash S^1$, $\mathbb{CP}^2 \backslash \{\infty\}$, $\mathbb{CP}^2 \backslash S^1$ which are not Kerr, Taub-bolt, or Chen-Teo, respectively?