Causal structure of spacetime and construction of quantum field theory

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dedicated to Christian Bär

Introduction

Quantum Field Theory yields an almost complete description of fundamental physics (up to gravity), but still suffers from severe mathematical problems.

Approaches from 2 sides:

- Construction of models
- Characterization by axioms

The construction of physically relevant models is up to now possible only in terms of uncontrolled approximations (formal perturbation theory, lattice models,. . .)

On the other hand, the axiomatic approach does not specify concrete models, but yields only general features (spin-statistics connection, PCT theorem, *etc*).

New approach: Add algebraic relations valid in formal perturbation theory to the axiomatic structure.

Input: Reformulation of perturbation theory

- Generalization to generic Lorentzian spacetimes
- Renormalization by conditions on S-matrices

Result: Construction of a functor from a category of spacetimes to the category of C*-algebras satisfying fundamental physical properties, in particular causality in two aspects:

Independence of causally disjoint regions and dependence in causally dependent regions.

At present: Existence of physically interesting representations open.

Framework:(Trivial) vector bundles over some globally hyperbolic Lorentzian manifold M, together with a Lagrangian density

$$L(\phi, d\phi) = \frac{1}{2}\langle d\phi, \wedge \star d\phi\rangle - V(\phi)d\mu \ , \ \phi \text{ smooth section}$$

Algebras of observables generated by local operations, understood as S-matrices induced by a local interaction,

$$S(F) = Te^{iF} \ , \ T \text{ time ordering operator}$$

$$F[\phi] = \sum \int \langle f_n(x), \phi(x)^{\otimes n}\rangle$$

local functional of smooth sections ϕ, with test densities f_n.

Relations

Causal factorization

$$S(F+G) = S(F)S(G) \ , \ \text{if } \mathrm{supp}F \cap J_-(\mathrm{supp}G) = \emptyset$$

J_- causal past of a region.

Unitary field equation

$$S(F) = S(F^\psi + \delta L(\psi))$$

ψ compactly supported, $F^\psi[\phi] = F[\phi+\psi]$,
$\delta L(\psi) = \int L(\phi+\psi, d(\phi+\psi)) - L(\phi, d\phi)$

Relative S-matrices

$$S_G(F) \doteq S(G)^{-1} S(F+G)$$

Interpretation: S-matrices under an additional interaction G, since S_G satisfies the analogous unitary field equation

$$S_G(F) = S_G(F^{\psi} + \delta L(\psi) + G^{\psi} - G)$$

Stable causal factorization: Causal factorization for relative S-matrices implied by

$$S(F+G+H) = S(F+G)\, S(G)^{-1}\, S(G+H)$$

$$\text{if } \mathrm{supp}F \cap J_{-}(\mathrm{supp}H) = \emptyset$$

Algebras of local observables

Definition

Let N be a globally hyperbolic subregion of the spacetime M. The algebra $\mathfrak{A}(N, L)$ of local observables in N for a theory with Lagrangian L is the C*-algebra freely generated by unitaries $S(F)$, F local functional with $\mathrm{supp} F \subset N$ modulo the relations

- Stable causal factorization
- Unitary field equation
- $S(F_c) = e^{ic}$ for constant functionals $F_c[\phi] = c$, $c \in \mathbb{R}$.

Theorem

Let L be of second order in ϕ. Then the subalgebra $\mathfrak{A}_1(N, L)$ generated by $S(F)$ with $F[\phi] = \int \langle f(x), \phi(x) \rangle$, f smooth density on N, is isomorphic to the Weyl algebra, i.e. the C-algebra generated by unitaries $W(f)$ with the Weyl relation*

$$W(f)W(g) = e^{-i/2\langle f,(\Delta_R - \Delta_A)g\rangle} W(f+g)$$

$\Delta_{R/A}$ retarded/advanced Green operator of the field equation associated to L.

Note that the canonical commutation relations are not imposed, but follow from causality.

Construction of the model of locally covariant QFT with Lagrangian L:

$\mathfrak{A}(L)$ functor from the category of globally hyperbolic spacetimes to the category of unital C*-algebras.

$$\mathfrak{A}(L) : M \mapsto \mathfrak{A}(L, M)$$

$\chi : N \to M$ isometric and causality preserving embedding

$$\mathfrak{A}(L)\chi : \begin{cases} \mathfrak{A}(L, N) & \to & \mathfrak{A}(L, M) \\ S(F) & \mapsto & S(\chi_* F) \end{cases}$$

with $\chi_* F[\phi] = F[\phi \circ \chi]$.

Renormalization group

The Weyl algebra $\mathfrak{A}_1(M,L)$ is simple and satisfies the time slice axiom

$$\mathfrak{A}_1(N,L) = \mathfrak{A}_1(M,L) \ , \ \ N \supset \Sigma \text{ Cauchy surface of } M,$$

but the full algebra $\mathfrak{A}(M,L)$ admits a large class of automorphisms which act trivially on the Weyl algebra and map local algebras into themselves. They are of the form

$$\beta_Z(S(F)) = S(Z(F))$$

where Z maps local functionals to local functionals, acts trivially on linear and constant functionals, preserves the support and satisfies the additivity relation

$$Z(F+G+H) = Z(F+G) - Z(G) + Z(G+H) \,,\ \mathrm{supp} F \cap \mathrm{supp} H = \emptyset$$

(this preserves the stable causality condition) and the relation

$$Z(F^{\psi} + \delta L(\psi)) = Z(F)^{\psi} + \delta L(\psi)$$

(this preserves the unitary field equation).

The group of invertible transformations Z may be considered as the

nonperturbative analogue of the renormalization group
in the the sense of Stückelberg-Petermann.

This group characterizes the ambiguity in the process of renormalization. It is related, but not identical to the Wilsonian concept of the renormalization group
(see Brunetti, Dütsch, F 2009).

Unitary Anomalous Master Ward Identity

We now consider symmetries of the configuration space $\mathcal{C}^\infty(M, \mathbb{R}^n)$ (linear field redefinition):

$$(x \mapsto \phi(x)) \longrightarrow (x \mapsto A(x)^t \phi(x))$$

with a smooth $\mathrm{GL}(n, \mathbb{R})$ valued function $A : x \mapsto A(x)$ with compact support ($\bullet^t$ denotes the transpose).

Then A induces a transformation A_* of functionals $A_* F(\phi) = F(A^t \phi)$ and of the Lagrangian, and

$$\alpha_A(S(F)) = S(A_* F)$$

defines an isomorphism $\mathfrak{A}(M, L) \to \mathfrak{A}(M, A_* L)$.

On the other hand, we can embed the algebra $\mathfrak{A}(M, A_* L)$ into $\mathfrak{A}(M, L)$ by considering $\delta_A L \doteq \int A_* L - L$ as an interaction,

$$S(F) \mapsto S(\delta_A L)^{-1} S(\delta_A L + F) \ .$$

Composing both maps

$$S(F) \mapsto S(\delta_A L)^{-1} S(\delta_A L + A_* F)$$

we obtain an automorphism of $\mathfrak{A}(M, L)$.

For the free Lagrangian, this automorphism acts trivially on $S(F)$ for linear functionals F.

We postulate the following relation: there exists a map ζ from the group generated by the field redefinitions A to renormalization group elements ζ_A such that for all local functionals F

$$S(\delta_A L + A_* F) = S(\zeta_A F)$$

(unitary anomalous Master Ward identity)

The map ζ has to satisfy the cocycle relation

$$\zeta_{AB} = \zeta_B \zeta_A^B$$

where the action of B on a renormalization group element Z is defined by

$$Z^B = B_L^{-1} Z B_L \ , \ B_L F = B_* F + \delta_B L \ .$$

A similar relation can be postulated for compactly supported diffeomorphisms χ of M.

They act on configurations by $\phi \mapsto \phi \circ \chi$ and on functionals F by

$$\chi_* F(\phi) = F(\phi \circ \chi) \ .$$

We set $\delta_\chi L = \int \chi_* L - L$ and require

$$S(\delta_\chi L + \chi_* F) = S(\zeta_\chi F)$$

for a cocycle ζ with values in the renormalization group.

This requires a slight generalization of the framework where also changes in the kinetic term of the Lagrangian are admitted as interactions.

Since this also changes the causal structure, the stable causality relation has to be modified to

$$S(F+G+H) = S(F+G)S(G)^{-1}S(G+H) \text{ if } \mathrm{supp}F \cap J_-^G(\mathrm{supp}H) = \emptyset$$

Here J_-^G is the past with respect to the causal structure with interaction G.

Note that this relation goes beyond perturbation theory.

Let $\mathcal{G}$ be the group generated by shifts ψ, linear field redefinitions A and diffeomorphisms χ, (affine bundle automorphisms) then the unitary Anomalous Master Ward Identity postulates the equation

$$S(\delta_g L + g_* F) = S(\zeta_g F) \ , \ g \in \mathcal{G}$$

for some cocycle ζ. This identity defines an ideal I_ζ of $\mathfrak{A}(M, L)$, with quotient $\mathfrak{A}(M, L, \zeta)$ and with local subalgebras $\mathfrak{A}(N, L, \zeta)$.

Theorem

The unitary Anomalous Master Ward Identity holds in formal perturbation theory where it is equivalent to the Anomalous Master Ward Identity of Brennecke-Dütsch (2008) and of the renormalized Quantum Master Equation of the Batalin-Vilkovisky formalism (F, Rejzner 2013).

Properties of the Haag-Kastler net $\mathfrak{A}_{M,L,\zeta} : N \mapsto \mathfrak{A}(N, L, \zeta)$:

- $\mathfrak{A}_{M,L,\zeta}$ satisfies the time slice axiom.
- Anomalous Noether Theorem:
  Symmetries of the Lagrangian give rise to unitaries which implement locally the symmetry, up to renormalization group transformations ("quantum symmetries").
- For vanishing ζ classical symmetries coincide with quantum symmetries.
- Symmetries of the Lagrangian induce a flow of theories which can be described in two equivalent ways:
  As an action of the symmetry on the cocyle ζ
  or as a flow of the Lagrangian, together with a transformation of observables (known as "running coupling constants")

Summary

The algebraic construction yields a definite algebraic quantum field theory which satisfies the axioms of locality, covariance and time-slice. It is fixed by two ingredients: a classical Lagrangian L and a cocycle ζ characterizing the anomalies. The cocycle relation corresponds to the Wess-Zumino consistency conditions of perturbation theory.

We restricted ourselves here to scalar fields. Fermionic fields can be treated similarly after adding auxiliary Grassmann parameters in a consistent way (BDFR21).

Perturbation theory delivers a nontrivial representation of a dense subalgebra in terms of formal power series of Hilbert space operators. In particular, the scaling anomaly for massless scalar theories and the axial anomaly for massless Fermi fields are recovered.

For the subalgebra generated by $S(F)$ with functionals F of second order in ϕ, the Fock space representation can be extended. This construction goes beyond formal perturbation theory, since changes of the causal structure are included. (Buchholz, F 2020)

Open problems:

Gauge theories are not yet included (work in progress).

As a C*-algebra, $\mathfrak{A}(M, L, \zeta)$ has a full state space and a faithful Hilbert space representation. But the structure of the state space, in particular its physical interpretation remains to be determined.

In our construction, ζ can be freely chosen. But according to Stora's Main Theorem of Renormalization, the equivalence class of the cocycle ζ should be fixed by the standard stability requirements, as *e.g.* existence of a vacuum and of KMS states, particle interpretation *etc.*

Conjecture: There is a distinguished equivalence class of cocycles which characterizes the spectrum condition.