Scalar curvature rigidity

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Among the classical curvatures, scalar curvature is the weakest one. As a consequence, it remains thus a challenging question whether a given manifold (possibly with boundary, with corners or singularities) carries a metric of positive or non-negative scalar curvature, assuming suitable behavior at the boundary or close to the singularities.

In the past the Atiyah-Singer index theorem provided a tremendous source for statements related to positive/nonnegative scalar curvature. Amazingly, the field got new impetus in the recent years, by focussing on new types of questions, and new applications of index theoretic methods, driven in particular by Gromov's article [15]. Other progress in the field is related to recent advances in general relativity.

In this seminar we focus on some chosen aspects in the field that are well accessible to young mathematicians, and that are thus well-suited for our block seminar.

1 Spin Geometry and Index Theory

We expect that the previous knowledge of our participants will vary considerably. The goal of this first part is to lay joint foundations for the following talks. Although we hope that this introduction may provide the most important techniques and result, the non-expert participants are strongly encouraged to read in the reference of the following talks prior to the seminar. This applies, in particular, to the topics presented of the first three talks. A detailed exposition of the content of these talks including proofs may easily fill a full semester lecture course, thus the talks can only give rough overviews which might be a bit short for students less familiar with spin geometry.

Further literature that might be helpful is contained in the books [18], [13], [24], [9], the overview article [16], and lecture notes by Bär [4], by Hanke , and by Ammann and Ludewig [2] and partial lecture notes by Ammann [1].

The purpose of this section implies that the current plan for these first five talks is still a bit preliminary as we might adapt it to the needs of the participants later.

1. Talk: Characteristic classes of vector bundles.

The speaker shall introduce characteristic classes of vector bundles, in particular, Pontrjagin classes, the Euler class, the $\hat{A}$-class, and – possibly – the $\mathcal{L}$-class in the signature theorem. A classical reference is [23, §1-15]. In view of the limited time, the speaker should focus on characterizing the main properties of these classes, rather trying to be logically as complete as possible. Both axiomatic properties such as the behaviour under pullback and the calculation of characteristic classes in terms of Chern-Weill theory is important. In view of the broad content, it might be wise to follow the relevant sections in [2] or [4, Section 4]. Additional literature may be found in [3], [18, Appendix B] and [9, Section 1.5].

2. Talk: Introduction to spin geometry. The topic of this talk are Clifford algebras, their representations, spinor bundles, Dirac operators and the Schrödinger-Lichnerowicz formula. We recommend to follow Sections 2.1 to 2.5 in [4], e.g. the Schrödinger-Lichnerowicz formula is Theorem 2.5.11 in there. Other literature is [18, I§1-§5, II §1- §5, and Appendix A] and [9, Sections 3.1 to 3.2]; these references provide deeper and more general insight, but are probably less suitable as a source for this talk.

3. Talk: Atiyah-Singer index theorem on compact manifolds without and with boundary. State and explain the Atiyah-Singer index theorem for closed manifolds [4, Theorem 4.3.7]. Introduce the Atiyah-Patodi-Singer (APS) boundary condition, see [22, around Eq. (In.8)] and [6, Example 7.6] (arxiv), and explain how the Atiyah-Singer index theorem generalizes to the Atiyah-Patodi-Singer index theorem for compact manifolds with boundary, see [22, Eq. (In.6)] . It would be helpful to say some words about local boundary conditions [6, Section 7.4] – the speaker might synchronize with the speaker of "5. Talk" to mainly specialize to the example required in that talk.
If time admits also give some application to positive scalar curvature, see e.g.[2, Section 4.3 and other parts of Chapter 4] or others according to the preferences of the speaker.

4. Talk: Llarrull type rigidity.
The goal of this talk is to present Llarull type rigidity following the work of Goette-Semmelmann: [14]. The central statement in this article is Theorem 0.2. In view of the following talks the most important case is that M is a round sphere, but it would be interesting to see the general result. For the following talks it is important to see the proof in the even-dimensional case and its relation to $\chi(M)$. If time admits, then also discuss the odd-dimensional case.
A new approach towards the odd-dimensional case is provided by Li, Su, and Wang [19], but is up to the speaker, whether he/she wants to incorporate these new ideas in the talk or whether the talks remains closer to [14].

5. Talk: Llarrull type rigidity on manifolds with boundary.
The goal of this talk is to present Llarull type rigidity following the work of Lott [20]. In this article Llarrull type rigidity theorems for manifolds with boundary are proved. If time is left, the speaker may also comment on how to apply similar techniques for proving statements about a spinorial quasi-local mass [21])

2 Rigidity for non-negative scalar curvature on polytopes

The aim for this part is to prove Brendle's scalar curvature rigidity theorem for polytopes : Take a compact convex polytope Ω in $\mathbb{R}^{n\geq 3}$ and a Riemannian metric g defined in a neighbourhood of it which as non-negative scalar curvature and with respect to which the boundary faces have non-negative mean curvature. If the dihedral angles of the boundary faces agree with the Euclidean ones, then g will be flat and the boundary faces totally geodesic.

6. Talk: The boundary value problem [12, §1–2]. Describe the main idea of the proof: Approximate Ω by smooth convex domains $\Omega_\lambda \subset \Omega$. Construct an m-tuple of harmonic spinors satisfying a certain local boundary condition on Ω_λ. As $\lambda \to \infty$ this tuple converges to a tuple of parallel spinors in the interior of Ω providing a parallel and orthogonal trivialization of the spinor bundle with respect to g. Hence g is flat.
Then set up the boundary value problem on Ω_λ in detail. Highlight the integral formula in Proposition 2.9 and its consequences. The formulation of the boundary condition and its ellipticity is the content of Proposition 2.14. Finish with the index computation in Proposition 2.15. In case the speaker wants to comment more on boundary conditions than the main reference for this talk, we reccomend the approach by Bär and Ballmann [6, 7].

7. Talk: Approximation by smooth domains [12, §3]. Decribe the approximation by smooth convex domains $\Omega_\lambda \subset \Omega$ in detail. Introduce the map N which is a deformation of the Euclidean Gauss map of $\Sigma_\lambda = \partial\Omega_\lambda$. Discuss Proposition 3.9 which provides a pointwise lower bound for the mean curvature H of Σ_λ in terms of the differential of N and a function V_λ. Pointwise and integral estimates for the negative part of V_λ are derived in the remainder of the section - give an overview!

8. Talk: The proof [12, §4]. Finish the proof of Brendle theorem using the results of the previous talks. State and use Corollary A.7 from the Appendix (A variant of a theorem of Fefferman and Phong). Comment on the proof only in case there is extra time. Discuss why the boundary faces become totally geodesic.

3 Scalar Curvature Rigidity of Warped Product Metrics

Gromov asked the following question: Suppose g_{S^n} is the standard metric on the phere S^n, assume that A is a "small" closed subset of S^n and g is Riemannian metric on $S^n \setminus A$, such that $g \geq g_{S^n}$ and such that the scalar curvature of g is at least $n(n-1)$ everywhere. Is the g the restriction of g_{S^n}. In the special case $A = \emptyset$, this is the Llarrull theorem for the sphere, discussed above. If A is a closed ball of radius $> \pi/2$, the answer to Gromov's question is negative, but one assume that the answer is yes for sufficiently small sets A. Without further assumtions, the strongest result so far is the article [8] where they prove this result for $A = \{p\}$ and $A = \{p, -p\}$. This is the topic of this part.

9. Talk: The holographic index theorem. In this talk we prove the holographic index theorem by Bär-Brendle-Hanke-Wang, given in Appendix B of [8]. This theorem relates the index on a manifold with boundary with that of the boundary. The proof of this theorem relies on the unique continuation property for the Dirac operator, see [10, Theorem 8.2] for a proof. If time remains, please comment on unique continuation for Dirac operators, and possibly sketch its proof.

10. Talk: A rigidtiy theorem for generalized cylinders. The goal of this talk is Theorem A in [8]. After some motivation and presentation of this theorem, the speaker should explain essential parts of its proof which is given in [8, Section 2]. For time reasons, it might be that the speaker is forced to black-box some more involved calculations as e.g. (parts of) Proposition 2.5. For the audience it is probably more important to understand the geometric role of some terms and their significance for estimates than a line-by-line verification. E.g. some terms correspond to the classical Weitzenböck formular in case $\tilde{M}$ has no boundary, and $\Psi \equiv 0$.
Note that in [8] the case is worked out, when the dimension n of the generalized cyinder is even. This case is more complicated than the case n odd, and relies on a "dirty trick" with a map $h : S^{n-1} \times S^1 \to S^n$. The proof for n odd is, is treated in Subsection 2.2, explaining how to modify the arguments before. It is up to the speaker (in synchronization with the organizers) to choose whether he/she wants to present the even-dimensional case (worked out in detail) or the simpler odd-dimensional setting one (where one should modify the even-dimensional arguments along the lines of Subsection 2.2).
Let us mention that the even-dimensional case can also be solved using an efficient spectral flow argument instead of the function h, see [5]. This point of view probably out of reach within the talk, but might encourage the speaker to concentrate on the case n odd in the seminar.

11. Talk: Larrull type rigidity on $S^n \setminus \{-p, p\}$. Building on the previous talk, the speaker should present and prove Theorem B of [8]. This is carried out in [8, Section 3]. The speaker should synchronize with the previous speaker whether they want to concentrate on the even- or odd-dimensional case. Unfortunately, the odd-dimensional setting is only briefly indicated, see Subsection 3.2, but we hope that the indications are sufficient for the speaker. If they are not, please present the case n even.

4 The harmonic function approach in general relativity

In the literature, there are many approaches to the positive mass theorem. Here, we discuss an approach for three-dimensional manifolds which uses spacetime harmonic functions of linear growth at infinity. This ansatz shares similarities with both the minimal hypersurface approach and the Witten approach with harmonic spinors. The level sets of the harmonic functions play a role similar to the Schoen-Yau minimal hypersurfaces and the Weizenböck formula for the harmonic spinor is replaced by a new identity for harmonic functions found by [25].

12. Talk: A level-set formula for harmonic functions [11, §1-4]. The proof positive mass theorem in dimenion three can be reduced to metrics on $\mathbb{R}^3$ which agree with the Schwarzschild metric outside a compact set. After discussing properties of linearly growing harmonic functions, a crucial identity is derived for the level sets of harmonic functions with Neumann boundary data on one boundary component and a nondegeneracy condition on the other component. If time permits, compare this formula to Stern's identity for S^1-valued harmonic maps [25].

13. Talk: The Schwarzschild approach for PMT [11, §5]. Using the aforementioned identity and the Schwarzschid, reduction, a proof of the positive mass theorem is given.

14. Talk: An alternative proof and the spacetime version of PMT [11, §6] and [17]. Instead of the Schwarzschild reduction, one can also use harmonic coordinates in order to derive the positive mass theorem from the level set identity. In a follow up paper [17], the approach was further developed to give a new proof of the positive energy theorem for initial data sets. Give a rough overview of that article.

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