Obstructions to the desingularization of nearly G_2 and nearly Kähler conifolds

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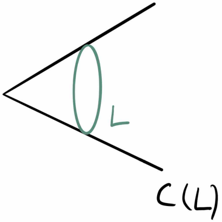
Geometry of cones and sine cones

(L^n, g_L)

Cone

$C(L) = (0,\infty) \times L$

$g_C = dr^2 + r^2 g_L$



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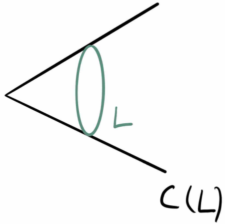
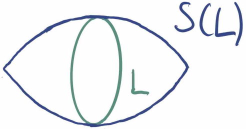


Sine cone

$S(L) = (0,\pi) \times L$

$g_S = dt^2 + \sin(t)^2 g_L$



Geometry of cones and sine cones

(L^n, g_L)

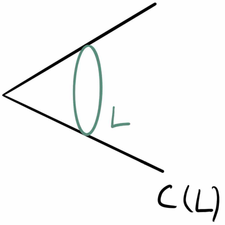
$\mathrm{Ric}_{g_L} = (n-1) g_L$

Cone

$C(L) = (0,\infty) \times L$

$g_C = dr^2 + r^2 g_L$

$\mathrm{Ric}_{g_C} = 0$



Sine cone

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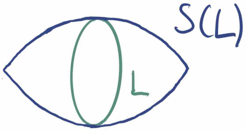
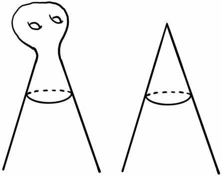
$\mathrm{Ric}_{g_S} = n\, g_S$



A construction method for positive Einstein manifolds?

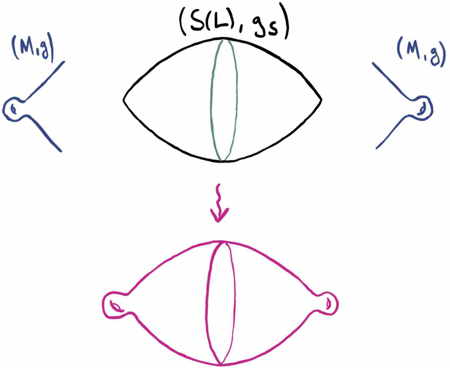
- (L, g_L) with $\mathrm{Ric}_{g_L} = (n-1)g_L$
- The cone $(C(L), g_C)$ is Ricci flat
- A **desingularization** of $(C(L), g_C)$ is a **Ricci flat, asymptotically conical manifold** (M, g) with **tangent cone** $(C(L), g_C)$



- $(M, \epsilon^2 g)$ converges to $(C(L), g_C)$ as $\epsilon \to 0$

A construction method for positive Einstein manifolds?

- (L, g_L) with $\mathrm{Ric}_{g_L} = (n-1)g_L$
- (M, g) Ricci flat and asymptotically conical with tangent cone $(C(L), g_C)$
- Glue in a copy of $(M, \epsilon^2 g)$ at both tips of $(S(L), g_L)$



- The closed manifold so constructed is Einstein away from the tips.
- **Question:** Can this be perturbed to an actual Einstein manifold?

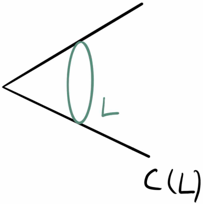
Geometry of cones and sine cones revisited

(L^n, g_L)

Cone

$C(L) = (0,\infty) \times L$

$g_C = dr^2 + r^2 g_L$



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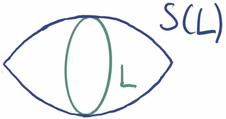


Geometry of cones and sine cones revisited

(L^n, g_L)

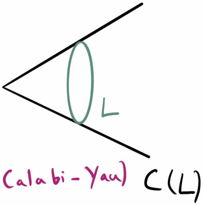
n=5: Sasaki-Einstein

Cone

$C(L) = (0,\infty) \times L$

$g_C = dr^2 + r^2 g_L$

Holonomy SU(3) (Calabi-Yau)



Sine cone

$S(L) = (0,\pi) \times L$

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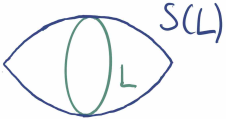
Nearly Kähler



Geometry of cones and sine cones revisited

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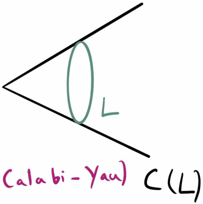


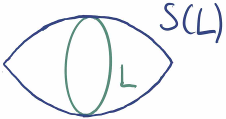
Holonomy $SU(3)$ (Calabi-Yau)

Holonomy G_2

(L^n, g_L)

$n=5$: Sasaki-Einstein

$n=6$: Nearly Kähler



Sine cone

$S(L) = (0,\pi) \times L$

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Nearly Kähler

Nearly G_2

Some G_2 geometry

- G_2 structures on a 7-manifold are parametrized by **stable 3-forms**
- For a stable 3-form $\varphi \in \Omega^3_+(M)$,

$$G_x = \{A \in \mathrm{Aut}(T_x M) : A^*\varphi(x) = \varphi(x)\}$$

  is isomorphic to $G_2 \subset SO(7)$
- φ induces a Riemannian metric g_φ
- $\mathrm{Hol}_0(M, g_\varphi)$ is isomorphic to a subgroup of G_2

$$\Updownarrow$$

$$d\varphi = 0, \qquad d *_{g_\varphi} \varphi = 0.$$

- (M, φ) is **nearly $\mathbf{G_2}$** if

$$d\varphi = 4 *_{g_\varphi} \varphi$$

Obstruction to desingularizing nearly G_2 sine cones

Theorem (S., 2022)

- *(L, g_L) 6-dimensional nearly Kähler manifold*
- *$(C(L), g_C, \varphi_C)$ the associated G_2 cone*
- *$(S(L), g_S, \varphi_S)$ the associated nearly G_2 sine cone*
- *(M, g, φ) is an AC G_2 manifold with tangent cone (C, g_C, φ_C)*

$$\exists \text{ smooth desingularization of } (S(L), \varphi_S) \text{ by } (M, \varphi)$$

$$\implies \qquad *\varphi \text{ is exact}$$

Idea of proof.

(X, φ_t), $t \in (0, \epsilon)$

$$\varphi_t \to \varphi_S \quad \text{as} \quad t \to 0, \qquad t^{-3}\varphi_t \to \varphi \quad \text{as} \quad t \to 0$$

Let $\widetilde{\varphi}_t = t^{-3}\varphi_t$, then $d\widetilde{\varphi}_t = 4t * \widetilde{\varphi}_t$. Apply $\frac{d}{dt}\big|_{t=0}$. □

The rate of an asymptotically conical manifold

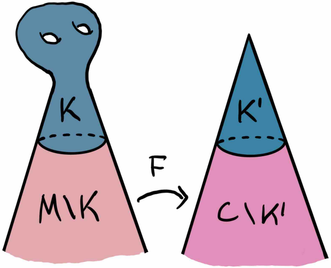
Definition

Let (C, g_C) be a cone.

A complete Riemannian manifold (M, g) is **asymptotically conical** with **tangent cone** (C, g_C) at **rate** $\nu < 0$, if

- $\exists$ compact set $K \subset M$, $K' \subset C$
- $\exists$ diffeomorphism $F : C \backslash K' \to M \backslash K$

such that $$r^k |\nabla^k (F^* g - g_C)|_{g_C} = O(r^\nu)$$



Topology of asymptotically conical G_2 manifolds

Theorem (S., 2022)

Let (M, φ, g) be an AC G_2 manifold with rate $\nu < -3$.

$$*\varphi \text{ is exact} \iff (M, g) \text{ is isometric to } (\mathbb{R}^7, g_{\mathrm{eucl}})$$

- Therefore, sine cones can not be desingularized by AC G_2 manifolds with rate $\nu < -3$
- The classical examples are: [Bryant and Salamon (1989)]

Manifold	**Rate**	$\mathbf{H^4_{dR}(M)}$
$\Lambda^2_- S^4$	-4	$\mathbb{R}$
$\Lambda^2_- \mathbb{CP}^2$	-4	$\mathbb{R}$
$\Sigma S^3 = \mathbb{R}^4 \times S^3$	-3	0

- There are infinitely many more examples of rate -3 [Foscolo, Haskins, Nordström 2021]

Idea of the proof

Theorem (S., 2022)

Let (M, φ, g) be an AC G_2 manifold with rate $\nu < -3$.

$$*\varphi \text{ is exact} \qquad \Longleftrightarrow \qquad (M, g) \text{ is isometric to } (\mathbb{R}^7, g_{\mathrm{eucl}})$$

- A G_2 cone carries the conformal Killing vector field $r\partial_r$ for which

$$\mathcal{L}_{r\partial_r} g_C = 2 g_C,$$

$$\mathcal{L}_{r\partial_r} \varphi_C = 3\varphi_C, \qquad \mathcal{L}_{r\partial_r} * \varphi_C = 4 * \varphi_C.$$

- Goal: construct “approximate” conformal Killing vector field on (M, g)
- $r\partial_r$ is the gradient vector field of $\frac{1}{2}r^2$
- There exists a function $u : M \to \mathbb{R}$ asymptotic to $\frac{1}{2}r^2$ and $\Delta u = -n$
- The form $\kappa = \mathcal{L}_{\mathrm{grad}\, u} * \varphi - 4 * \varphi$ is harmonic
- Asymptotics of u and Hodge theory imply κ vanishes
- The only complete cone is $\mathbb{R}^n$ with the Euclidean metric

The nearly Kähler setting

Theorem (S., 2022)

Suppose (L, g_L) is a Sasaki–Einstein 5-manifold and (M, g) is an asymptotically conical Calabi–Yau manifold with tangent cone $(C(L), g_C)$. Then if the rate of (M, g) is smaller than -3, the sine cone $(S(L), g_S)$ can not be smoothly desingularized by gluing in (M, g).

Theorem (S., 2022)

$$(M, \underbrace{\omega \in \Omega^2(M)}_{\text{Kähler form}}, \underbrace{\Omega \in \Omega^3(M, \mathbb{C})}_{\text{holomorphic volume form}}, g) \text{ AC CY 6-manifold, rate } \nu < -3$$

Then:

$$\omega^2 \text{ and } \operatorname{Re}\Omega \text{ are both exact}$$

$$\Updownarrow$$

$$(M, g) \text{ is isometric to } (\mathbb{R}^6, g_{\mathrm{eucl}}).$$