Calibrations and energy-minimizing mappings of rank-1 symmetric spaces

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Energy of mappings of Riemannian manifolds

Let (M^m, g) and (N^n, h) be Riemannian manifolds:

Definition

The **energy** of a mapping $F : (M, g) \to (N, h)$ is

$$E_2(F) = \int_M |dF|^2 dVol_M, \qquad (1)$$

where $|dF|$ is the Euclidean norm of the differential dF.

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For $F : (M, g) \to \mathbb{R}$, this is the Dirichlet energy of the function F, and when $dim(M) = 1$, this coincides with the standard definition of the energy of a path in a Riemannian manifold.

N.B. Many authors define the energy to be $\frac{1}{2}$ the definition above.

Theorem (H. [Hois21])

Let $\widetilde{g}$ be a Kähler metric on $\mathbb{C}P^2$, and let $F : (\mathbb{C}P^2, \widetilde{g}) \to (M, g)$ be a mapping to a Riemannian manifold.

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$$inf\{|f(S^2)| \;\Big|\; f : S^2 \to (M, g) \text{ is in the free homotopy class of } F|_{\mathbb{C}P^1}.\}$$

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Moreover, $F(\mathscr{V})$ is minimal in (M, g) and its second fundamental form can be diagonalized by a unitary basis rel. to F^*g.

Harmonic mappings

Eells and Sampson [ES64] initiated the study of the critical points of the energy functional on the space of mappings between compact Riemannian manifolds:

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Theorem

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Eells, Sampson [ES64]: *Every mapping $f : (M, g_M) \to (N, g_N)$ is homotopic to a harmonic mapping f_∞ with $E_2(f_\infty) \leq E_2(f)$.*

The results of [ES64] and subsequent work of Hartman [Har67] imply that the identity mapping of a closed Riemannian manifold with non-positive sectional curvature is energy minimizing in its homotopy class.

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- Conformal dilations of the round sphere (S^n, g_0), $n \geq 3$ give energy-decreasing deformations of the identity mapping. In fact, the energy of these mappings decays to 0.

  In particular, the identity mapping of (S^n, g_0), $n \geq 3$ is not a stable critical point of the energy functional and the identity map is not energy-minimizing in its homotopy class.

Harmonic and energy minimizing Maps

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- Brian White has proven:

Theorem (White [Wh86])

Let (M, g) be a closed Riemannian manifold with $\pi_1(M) = \pi_2(M) = 0$. Then the identity mapping of M is homotopic to maps with arbitrarily small energy.

In fact, White's results say more – more on this later.

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Theorem (Croke [Cr87])

Let $F : (\mathbb{R}P^n, g_0) \to (M, g)$ be a mapping to a Riemannian manifold (M, g).

Let $L^\star$ be the infimum of the lengths of paths in the free homotopy class of $F(\gamma)$, where γ is a geodesic of $(\mathbb{R}P^n, g_0)$. Then:

Energy minimizing mappings of real projective space

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$$E_2(F) \geq \frac{\sigma(n)}{2\pi^2} {L^\star}^2, \qquad (2)$$

where $\sigma(n) = Vol(S^n)$. Equality implies that F is a homothety onto a totally geodesic submanifold.

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where $\sigma(n) = Vol(S^n)$. Equality implies that F is a homothety onto a totally geodesic submanifold.

In particular, the identity mapping of $(\mathbb{R}P^n, g_0)$ is energy minmimizing in its homotopy class, uniquely up to isometries.

The identity mapping of complex projective space with its canonical metric is also energy-minimizing in its homotopy class.

This follows from a much more general theorem of Lichnerowicz:

Theorem (Lichnerowicz [Lic70])

Let (X, h_X) *and* (Z, h_Z) *be compact Kähler manifolds.*

Let $F : X \to Z$ *be a mapping which is (anti-)holomorphic.*

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Let $F : X \to Z$ be a mapping which is (anti-)holomorphic.

Then for any mapping $f : X \to Z$ which is homotopic to F, $E_2(f) \geq E_2(F)$. Equality holds iff f is also (anti-)holomorphic.

In particular, holomorphic and antiholomorphic mappings of compact Kähler manifolds minimize energy in their homotopy classes.

So, to recap:

The identity mapping of $(\mathbb{R}P^n, g_0)$ is energy minimizing in its homotopy class. This follows from a lower bound for the energy of a mapping of $(\mathbb{R}P^n, g_0)$ with a rigid equality characterization (homothety onto a totally geodesic submanifold).

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The identity mapping of $(\mathbb{C}P^N, g_0)$ is energy minimizing in its homotopy class. This follows from Lichnerowicz's theorem about mappings of Kähler manifolds.

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The identity mapping of $(\mathbb{C}P^N, g_0)$ is energy minimizing in its homotopy class. This follows from Lichnerowicz's theorem about mappings of Kähler manifolds.

However, Lichnerowicz's theorem also implies that the equality case for $(\mathbb{C}P^N, g_0)$ must be bigger than for $(\mathbb{R}P^n, g_0)$:

There are biholomorphic mappings of $\mathbb{C}P^N$ in the homotopy class of the identity which are not isometries of g_0. Lichnerowicz's theorem implies these are also energy minimizing.

Calibrations and energy minimizing mappings of complex projective spaces

To recap:

The energy minimizing property of the identity mapping of $(\mathbb{C}P^N, g_0)$ can also be proven by an adaptation of Croke's proof for $(\mathbb{R}P^n, g_0)$. This also gives a lower bound for the energy of any mapping from $(\mathbb{C}P^N, g_0)$ to a Riemannian manifold (M, g):

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Let $F : (\mathbb{C}P^n, g_0) \to (M, g)$ be a mapping to a Riemannian manifold (M, g). Let $A^\star$ be the infimum of the areas of $f : S^2 \to M$ in the free homotopy class of $F(\mathbb{C}P^1)$. Then:

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Equality holds in (3) for the identity mapping, however the calibrated structure of a Kähler manifold implies that equality also holds for any holomorphic mapping $F : (\mathbb{C}P^N, g_0) \to (X, h)$ to a compact Kähler manifold.

The theorem at the beginning implies that holomorphic (and antiholomorphic) mappings are essentially the only such energy-minimizing mappings of $(\mathbb{C}P^2, g_0)$.

Moreover, it shows that the same sharp lower bound holds for the energy of mappings of $\mathbb{C}P^2$ with any Kähler metric $\widetilde{g}$, and that holomorphic and antiholomorphic mappings are the only such energy-minimizing mappings of Kähler metrics on $\mathbb{C}P^2$.

The proof of the the theorem at the beginning uses the following lemma:

Lemma (H. [Hois21])

Let (X, h) *be a Hermitian surface (of complex dimension* 2*).*

Suppose that for all $x_0 \in X$ *and all complex lines* Π *in* $T_{x_0}X$ *there is a complex curve* $\Sigma_\Pi \subseteq X$ *(of complex dimension* 1*) with* Π *tangent to* Σ_Π*, and with the mean curvature of* Σ_Π *vanishing at* x_0*.*

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In fact, a more general statement holds.

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Theorem (H. [Hois21])

Let $\widetilde{g}$ be a Kähler metric on $\mathbb{C}P^N$, suppose WLOG that $\widetilde{g}$ is cohomologous to the canonical metric g_0 with sectional curvature K satisfying $1 \leq K \leq 4$.

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Moreover, $F(\mathscr{V})$ is a minimal submanifold of (M, g), and the second fundamental form of $F(\mathscr{V})$ in (M, g) can be diagonalized by a unitary basis of F^*g.

In particular, for all complex surfaces Z in $\mathbb{C}P^N$, $F^*g|_{Z \cap \mathscr{V}}$ is a Kähler metric.

The systolic inequality for $\mathbb{R}P^2$ states:

Theorem (Pu [Pu52])

Let g be a Riemannian metric on $\mathbb{R}P^2$. Let A be its area, and let $sys(g)$ be the length of the shortest non-contractible curve in $(\mathbb{R}P^2, g)$. Then:

$$A \geq \frac{2}{\pi} sys(g)^2. \tag{5}$$

Equality holds if and only if g has constant curvature.

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Pu's inequality for $\mathbb{R}P^2$ can be seen as a special case of Croke's lower bound for the energy of $f : (\mathbb{R}P^2, g_0) \to (M, g)$:

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Let g be a Riemannian metric on $\mathbb{R}P^2$. Let A be its area, and let $sys(g)$ be the length of the shortest non-contractible curve in $(\mathbb{R}P^2, g)$. Then:

$$A \geq \frac{2}{\pi} sys(g)^2. \tag{5}$$

Equality holds if and only if g has constant curvature.

Pu's inequality for $\mathbb{R}P^2$ can be seen as a special case of Croke's lower bound for the energy of $f : (\mathbb{R}P^2, g_0) \to (M, g)$:

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Croke's lower bound for energy then gives Pu's inequality (5).

The stable systolic inequality for $\mathbb{C}P^N$ states:

Theorem (Gromov [Gr81], see also [BKSW09])

Let g be a Riemannian metric on $\mathbb{C}P^N$. Let $Vol(\mathbb{C}P^N, g)$ be its volume and $stsys_2(g)$ its stable 2-systole, that is:

$$stsys_2(g) = \lim_{k \to \infty} \frac{\mu_k}{k},$$

where μ_k is the minimum mass of an integral current representing $k \in H_2(\mathbb{C}P^N; \mathbb{Z}) \cong \mathbb{Z}$.

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$$Vol(\mathbb{C}P^N, g) \geq \frac{stsys_2(g)^N}{N!}.$$

Equality holds for the canonical metric g_0 on $\mathbb{C}P^N$.

However this implies that equality holds for all Kähler metrics $\widetilde{g}$ on $\mathbb{C}P^N$. Again, this follows from the calibrated structure of the Kähler metric.

Parallels with systolic geometry

Quaternionic projective space $\mathbb{H}P^N$ satisfies a stable systolic inequality like $\mathbb{C}P^N$.

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Theorem (Bangert, Katz, Shnider, Weinberger [BKSW09])

Let g be a Riemannian metric on $\mathbb{H}P^2$ and $stsys_4(g)$ the stable 4-systole of g (defined as above for $H_4(\mathbb{H}P^N;\mathbb{Z}) \cong \mathbb{Z}$). There is a positive constant D_2, independent of g, such that:

$$Vol(\mathbb{H}P^2, g) \geq D_2 stsys_4(g)^2. \tag{6}$$

The optimal constant in (6) satisfies $\frac{1}{6} \geq D_2 \geq \frac{1}{14}$, which excludes the value $\frac{3}{10}$ of the canonical metric.

Energy Functionals of Mappings of Riemannian Manifolds

The energy of a mapping belongs to 1-parameter family of invariants:

Let (M,g) and (N,h) be Riemannian manifolds:

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In particular, the identity mapping of $(\mathbb{H}P^N,g_0)$ is not 4-energy minimizing in its homotopy class.

More generally, Wei's results imply that the identity mapping of $(\mathbb{H}P^N,g_0)$ is an unstable critical point of the p-energy for $1\leq p<2+\frac{4N}{N+1}$ and a stable critical point for $p\geq 2+\frac{4N}{N+1}$.

The identity mapping of $(\mathbb{H}P^N, g_0)$ is not 4-energy minimizing in its homotopy class. However we have:

Theorem (H. [Hois21])

Let $(\mathbb{H}P^N, g_0)$ be the quaternionic projective space with its canonical Riemannian metric, with $N \geq 2$, and $F : (\mathbb{H}P^N, g_0) \to (M, g)$ a non-constant mapping to a Riemannian manifold.

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Let $B^\star$ be the minimum mass of an integral 4-current representing the class of $F_*(\mathbb{H}P^1)$ in $H_4(M;\mathbb{Z})$.

Then for all $p \geq 4$,

$$E_p(F) > \frac{\pi^{2N}}{(2N+1)!} (K_N B^\star)^{\frac{p}{4}} . \tag{7}$$

In (7), $K_N = \frac{32N^2(2N+1)}{\pi^2(2N-1)}$.

Some Questions

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  (The identity mapping of any closed Riemannian n-manifold minimizes p-energy in its homotopy class for $p \geq n$.)
- How can we characterize p-energy minimizing, or approximately p-energy minimizing, families of maps of $(\mathbb{H}P^N, g_0)$?

Thank you!

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