Two rigidity results for stable minimal hypersurfaces

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Outline:

- Introduction: minimal surfaces in $\mathbb{R}^3$ and the Bernstein problem.
- The stable Bernstein problem.
- A result of Schoen-Simon-Yau (1975).
- The stable Bernstein problem in $\mathbb{R}^4$: a new proof.
- A second result on finite index minimal hypersurfaces in positively curved closed manifolds.

Bibliography:

- G. C., P. Mastrolia, A. Roncoroni, *Two rigidity results for stable minimal hypersurfaces,* 2022 **submitted**.

Minimal surface: $M^2 \subset \mathbb{R}^3$ is a critical point of the area functional $\mathcal{A}_t$, for all compactly supported variations, i.e.

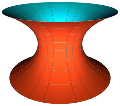
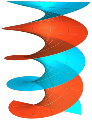
$$\frac{d}{dt}\Big|_{t=0} \mathcal{A}_t = 0\,.$$

Equivalently:

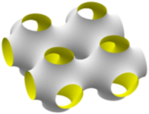
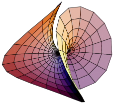
- M^2 is minimal $\Longleftrightarrow$ the mean curvature $H \equiv 0$;
- M^2 is minimal $\Longleftrightarrow$ M^2 can be expressed, locally, as the graph $\Gamma(u)$, where u solves the *minimal surfaces equation*:

$$\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0\,.$$

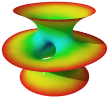
Classical examples: planes, catenoids, elicoids.

19th century examples (golden age): Schwarz minimal surfaces, Enneper surface, Henneberg surface, Bour's minimal surface, Neovius surface.

Modern examples: Gyroid, Costa's minimal surface, Chen-Gackstatter surface.



Theorem [Bernstein, 1914]
Let $u \in C^2(\mathbb{R}^2)$ be a solution of

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 \quad \textit{in } \mathbb{R}^2 .$$

Then u is an affine function, i.e.

$$u(x,y) = \alpha x + \beta y + \gamma ,$$

for some $\alpha, \beta, \gamma \in \mathbb{R}$.

Equivalently

An entire (i.e. defined on the whole plane $\mathbb{R}^2$) minimal graph in $\mathbb{R}^3$ is necessarily a plane.

The proof is based on a **Liouville-type theorem** for elliptic (not uniformly elliptic) operators, which holds true only in dimension 2.

Bernstein problem in higher dimension: an entire minimal graph in $\mathbb{R}^{n+1}$ is necessarily a hyperplane?

It resisted for a half-century and was solved thanks to the combined effort of:

- *Fleming (1965)*: new proof in the case $n = 2$;
- *De Giorgi (1965)*: $n = 3$;
- *Almgren (1966)*: $n = 4$;
- *Simons (1968)*: $n \leq 7$;

Their proofs are not based on Liouville-type theorems but on tools in **Geometric Measure Theory.**

Moreover

- *Bombieri-De Giorgi-Giusti (1969)*: for $n \geq 8$, there are minimal entire graphs that are not hyperplanes.

Two remarks:

- A minimal graph is **area-minimizing**: i.e. it is not only a critical point of the area functional, but also a minimum.

  This is not true for general minimal hypersurfaces (that are not graphic).
- Area-minimizing implies **stability**, that is

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathcal{A}_t \geq 0\,,$$

  for all compactly supported variations.

Stable Bernstein problem: if $M^n \hookrightarrow \mathbb{R}^{n+1}$ is a complete, orientable, immersed, **stable minimal** hypersurface, does M have to be necessarily a hyperplane?

Stable Bernstein problem: if $M^n \hookrightarrow \mathbb{R}^{n+1}$ is a complete, orientable, immersed, **stable minimal** hypersurface, does M have to be necessarily a hyperplane?

- **True** if $n = 2$: *do Carmo-Peng (1979).*, *Fischer-Colbrie-Schoen (1980)*, *Pogorelov (1981)*;
- **False** if $n \geq 7$: if $n \geq 8$, *Bombieri-De Giorgi-Giusti (1969)* constructed non-flat, orientable, complete, stable, minimal hypersurfaces (graphs) in $\mathbb{R}^{n+1}$, $n \geq 8$. Moreover, if $n \geq 7$, there exist non-flat area minimizing (not graphs) smooth hypersurfaces constructed by *Hardt-Simon (1985)*.
- Other dimensions? Schoen-Simon-Yau (1975): **true** if $n \leq 5$ and **Euclidean volume growth assumption**

$$\mathrm{Vol}(B_R) \leq CR^n \,.$$

- **Further interesting/partial results:** *Schoen-Yau (1979)*, *do Carmo-Peng (1982)*, *Bérard (1991)*, *Palmer (1991)*, *Miyaoka (1993)*, *Tanno (1996)*, *Cao-Shen-Zhu (1997)*, *Shen-Zhu (1998)*, *Chen (2001)*, *Nelli-Soret (2007)*.
- **True** if $n = 3$: *Chodosh-Li (2021)* and *C., Mastrolia, Roncoroni (2022)*.

Theorem [Chodosh-Li (2021) and C.-Mastrolia-Roncoroni (2022)]
A complete, orientable, immersed, stable minimal hypersurface $M^3 \hookrightarrow \mathbb{R}^4$ is a hyperplane.

Stable minimal hypersurface: M is *minimal* iff the mean curvature $H \equiv 0$; in this case we say that M is *stable* if

$$\int_M |A|^2 \varphi^2 \leq \int_M |\nabla \varphi|^2 \,, \qquad \text{for all } \varphi \in C_0^\infty(M)$$

where $A = A_M$ denotes the second fundamental form of M.

Remark: stability means non-negativity of the second variation, or, equivalently, non-positivity of the so-called *Jacobi operator*

$$\Delta + |A|^2 \,.$$

Theorem [Chodosh-Li (2021) and C.-Mastrolia-Roncoroni (2022)]
A complete, orientable, immersed, stable minimal hypersurface $M^3 \hookrightarrow \mathbb{R}^4$ is a hyperplane.

Main idea: use a **weighted volume comparison** for a suitable **conformal metric**, together with a new **weighted integral estimate**.

Inspiring papers: *Schoen-Simon-Yau (1975)*, *Fischer-Colbrie (1985)*, *Elbert-Nelli-Rosenberg (2007)*.

Before going into the proof, I will give you a 3-slides complete proof of *Schoen-Simon-Yau (1975)* result on stable minimal hypersurface $M^n \hookrightarrow \mathbb{R}^{n+1}$, $n \leq 5$, satisfying the volume assumption

$$\mathrm{Vol}(B_R) \leq CR^n \,.$$

Take a complete, orientable, immersed, stable minimal hypersurface $M^n \hookrightarrow \mathbb{R}^{n+1}$. We have

$$\int_M |A|^2 \varphi^2 \leq \int_M |\nabla \varphi|^2, \qquad \text{for all } \varphi \in C_0^\infty(M).$$

We test it with $\varphi = |A|^{1+q}\psi$, $q \geq 0$, with $\psi \in C_0^\infty(M)$, obtaining

$$\int_M |A|^{4+2q}\psi^2 \leq [(1+q)^2 + \varepsilon] \int_M |A|^{2q}|\nabla |A||^2 \psi^2 + \frac{1+q}{\varepsilon} \int_M |A|^{2+2q}|\nabla \psi|^2,$$

for every $\varepsilon > 0$, where we used Young's inequality. On the other hand, using the Codazzi equation

$$\nabla_k A_{ij} = \nabla_j A_{ik}$$

and Gauss equation, we get the well known Simons' identity for minimal hypersurfaces in $\mathbb{R}^{n+1}$

$$\frac{1}{2}\Delta |A|^2 = |\nabla A|^2 - |A|^4.$$

Moreover, since A is a Codazzi tensor, we have the improved Kato's inequality

$$|\nabla A|^2 \geq \frac{n+2}{n} |\nabla |A||^2.$$

Combining these, we obtain

$$|A|\Delta |A| + |A|^4 \geq \frac{2}{n} |\nabla |A||^2.$$

Multiplying it by $|A|^{2q}\psi^2$ and integrating by parts, we get

$$\left(\frac{2}{n}+1+2q-\varepsilon\right)\int_M |A|^{2q}|\nabla|A||^2\psi^2 \leq \int_M |A|^{4+2q}\psi^2 + \frac{1}{\varepsilon}\int_M |A|^{2+2q}|\nabla\psi|^2$$

for every $\varepsilon > 0$, where we used again Young's inequality. Since $q \geq 0$, for $\varepsilon > 0$ sufficiently small, combining these two estimates we obtain

$$\left\{1-[(1+q)^2+\varepsilon]\left(\frac{2}{n}+1+2q-\varepsilon\right)^{-1}\right\}\int_M |A|^{4+2q}\psi^2 \leq C\int_M |A|^{2+2q}|\nabla\psi|^2.$$

Let $q := \frac{p-4}{2}$. For $\varepsilon > 0$ small enough, we have

$$1-[(1+q)^2+\varepsilon]\left(\frac{2}{n}+1+2q-\varepsilon\right)^{-1} > 0$$

if $p \in [4, 4+\sqrt{8/n}]$ and we finally obtain

$$\int_M |A|^p\psi^2 \leq C\int_M |A|^{p-2}|\nabla\psi|^2 \qquad \forall\psi \in C_0^\infty(M).$$

Taking $\psi = \phi^{p/2}$, by Holder's inequality, we get

$$\int_M |A|^p\phi^p \leq C\int_M |\nabla\phi|^p \qquad \forall\phi \in C_0^\infty(M)$$

for all $p \in [4, 4+\sqrt{8/n}]$.

$$\int_M |A|^p \phi^p \leq C \int_M |\nabla \phi|^p \qquad \forall \phi \in C_0^\infty(M)$$

for all $p \in [4, 4 + \sqrt{8/n}]$. In particular, if $n \leq 5$, we can take $p = 5 + \delta$, for some $\delta > 0$ small. Let $x_0 \in M^n$, and let r denotes the distance function from x_0. We choose

$$\phi := \eta(r)\,,$$

where $0 \leq \eta \leq 1$, $\eta = 1$ on $[0, R]$, $\eta = 0$ on $[2R, \infty)$ and $|\eta'| \leq \frac{C}{R}$, for some $C, R > 0$. Plugging in the previous estimate, form every $R > 0$, we obtain

$$\int_M |A|^{5+\delta} \eta^{5+\delta} \leq C \int_{B_{2R} \setminus B_R} |\nabla \eta|^{5+\delta} \leq \frac{C}{R^{5+\delta}} \mathrm{Vol}(B_{2R}) \leq \frac{C}{R^\delta}$$

where we used the **Euclidean volume growth** assumption. Since $\delta > 0$, letting $R \to \infty$, we get

$$|A| \equiv 0 \quad \text{on } M^n,$$

and this concludes the proof. □

Theorem [Chodosh-Li (2021) and C.-Mastrolia-Roncoroni (2022)]
A complete, orientable, immersed, stable minimal hypersurface $M^3 \hookrightarrow \mathbb{R}^4$ is a hyperplane.

Idea of the proof:

- **Step 1:** construct $\widetilde{g}$ conformal to g such that

$$\mathrm{Ric}^{2,f}_{\widetilde{g}} := \mathrm{Ric}_{\widetilde{g}} + \nabla^2_{\widetilde{g}} f - \frac{1}{2} df \otimes df \geq 0\,.$$

- **Step 2:** prove that $\widetilde{g}$ is complete.
- **Step 3:** we obtain a weighted Bishop-Gromov volume estimate:

$$\mathrm{Vol}_f\left(B^{\widetilde{g}}_R\right) := \int_{B^{\widetilde{g}}_R} e^{-f}\, dV_{\widetilde{g}} \leq CR^{3+2} = CR^5\,. \tag{1}$$

- **Step 4:** prove a weighted integral estimate as in SSY
- **Step 5:** use a cutoff function $\phi = \eta(\widetilde{r})$ to conclude that

$$|A| \equiv 0 \quad \text{on } M^3.$$

It is well-known that the stability of $M^n \hookrightarrow \mathbb{R}^{n+1}$ implies the existence of $0 < u \in C^\infty(M)$ satisfying

$$-\Delta_g u = |A|_g^2 u \quad \text{in } M,$$

where g denotes the induced metric on M.

Let $k > 0$ and consider the **conformal metric**

$$\widetilde{g} = u^{2k} g\,.$$

Lemma

Let $f = k(n-2)\log u$. Then the Ricci tensor of the metric $\widetilde{g}$ satisfies

$$\mathrm{Ric}_{\widetilde{g}} + \nabla^2_{\widetilde{g}} f - \frac{1-k(n-2)}{k(n-2)^2} df \otimes df \geq \left(k - \frac{n-1}{n}\right) |A|_g^2 g\,.$$

In particular, if $n = 3$ and $k = \frac{2}{3}$, then the 2-Bakry-Emery-Ricci tensor satisfies

$$\mathrm{Ric}^{2,f}_{\widetilde{g}} := \mathrm{Ric}_{\widetilde{g}} + \nabla^2_{\widetilde{g}} f - \frac{1}{2} df \otimes df \geq 0\,.$$

Sketch of the proof: since $f = k(n-2)\log u$, we have

$$df = k(n-2)\frac{du}{u} \quad \text{and} \quad \nabla^2_g f = k(n-2)\left(\frac{\nabla^2_g u}{u} - \frac{du \otimes du}{u^2}\right),$$

which implies

$$\Delta_g f = k(n-2)\left(\frac{\Delta_g u}{u} - \frac{|\nabla_g u|^2_g}{u^2}\right).$$

On the other hand, from the standard formulas for a conformal change of the metric $\widetilde{g} = e^{2\varphi}g$, $0 < \varphi \in C^\infty(M)$, we get

$$\mathrm{Ric}_{\widetilde{g}} = \mathrm{Ric}_g - (n-2)\left(\nabla^2\varphi - d\varphi \otimes d\varphi\right) - \left[\Delta_g\varphi + (n-2)|\nabla_g\varphi|^2_g\right] g\,,$$

and

$$\nabla^2_{\widetilde{g}} f = \nabla^2_g f - (df \otimes d\varphi) + g(\nabla_g f, \nabla_g \varphi) g\,.$$

In our case $\varphi = k\log u$ and u solves $-\Delta_g u = |A|^2_g u$, thus

$$\mathrm{Ric}_{\widetilde{g}} + \nabla^2_{\widetilde{g}} f = \mathrm{Ric}_g - \frac{df \otimes df}{n-2} + k|A|^2_g g + \frac{|\nabla_g f|^2_g}{k(n-2)} g\,.$$

From Cauchy-Schwarz inequality we have

$$|\nabla_g f|_g^2 g \geq df \otimes df$$

and from Gauss equation in the minimal case we know that

$$\mathrm{Ric}_g = -A^2$$

and since A is traceless we also have the inequality

$$A^2 \leq \frac{n-1}{n} |A|_g^2 g \,.$$

Substituting in

$$\mathrm{Ric}_{\widetilde{g}} + \nabla^2_{\widetilde{g}} f = \mathrm{Ric}_g - \frac{df \otimes df}{n-2} + k|A|_g^2 g + \frac{|\nabla_g f|_g^2}{k(n-2)} g \,,$$

we conclude

$$\mathrm{Ric}_{\widetilde{g}} + \nabla^2_{\widetilde{g}} f - \frac{1-k(n-2)}{k(n-2)^2} df \otimes df \geq \left(k - \frac{n-1}{n}\right) |A|_g^2 g \,.$$

□

From now on, we take

$$n = 3\,, \quad k = \frac{2}{3}\,,$$

and $0 < u \in C^\infty(M)$ solution of

$$-\Delta_g u = |A|_g^2 u \quad \text{in } M.$$

Lemma
The metric $\widetilde{g} = u^{\frac{4}{3}} g$ is complete.

Sketch of the proof: as shown in *Fischer-Colbrie (1985)*, one can construct a minimizing geodesic in the metric $\widetilde{g} = u^{2k} g$, $\gamma = \gamma(s)$, where s is the $g-$arclength. By construction, the completeness of $\widetilde{g}$ is equivalent to prove that γ has infinite $\widetilde{g}-$length, i.e.

$$\int_\gamma d\widetilde{s} = \int_0^{+\infty} u^k(\gamma(s))\, ds = +\infty\,.$$

Since γ is minimizing, by the second variation formula

$$\int_0^\infty \left[(n-1)(\varphi_{\widetilde{s}})^2 - \widetilde{R}_{11}\varphi^2\right] d\widetilde{s} \geq 0\,, \quad \forall\, \varphi \in C_0^\infty(0,+\infty)$$

where $\widetilde{R}_{11}$ denotes the $\widetilde{g}-$Ricci curvature in the direction $\gamma_{\widetilde{s}}$.

Using the formula for the conformal change of the Ricci curvature, some integration by parts (cfr. *Elbert-Nelli-Rosenberg (2007)*) and controlling the $|A|^2$ terms, we get

$$(n-1)\int_0^{+\infty}(\varphi_s)^2 u^{-k}\,ds \geq 2k(n-2)\int_0^{+\infty}\varphi\varphi_s u^{-k-1}u_s\,ds + k\,[1-k(n-2)]\int_0^{+\infty}\varphi^2 u^{-k-2}(u_s)^2\,ds\,,$$

for $n \geq 3$, for every $\varphi \in C_0^\infty(0,+\infty)$ and for every $k \geq \frac{n-1}{n}$.
Now we choose $\varphi = u^k\psi$, with $\psi \in C_0^\infty(0,+\infty)$, we apply Young inequality, we manipulate and we obtain, assuming $k < 1$

$$0 \leq \left[\frac{k(t-1)^2}{1-k} - 2t + (n-1)\right]\int_0^{+\infty} u^k(\psi_s)^2\,ds - 2t\int_0^{+\infty} u^k\psi\psi_{ss}\,ds,$$

for every $t > 1$. If $n = 3$, $k = \frac{n-1}{n} = \frac{2}{3}$ and $t = \frac{3}{2}$ we have that

$$\frac{k(t-1)^2}{1-k} - 2t + (n-1) < 0\,,$$

hence,

$$0 \leq -\int_0^{+\infty} u^{\frac{2}{3}}(\psi_s)^2\,ds - 6\int_0^{+\infty} u^{\frac{2}{3}}\psi\psi_{ss}\,ds\,, \quad \forall\,\psi \in C_0^\infty(0,+\infty).$$

Hence,

$$0 \leq - \int_0^{+\infty} u^{\frac{2}{3}} (\psi_s)^2 \, ds - 6 \int_0^{+\infty} u^{\frac{2}{3}} \psi \psi_{ss} \, ds \,, \quad \forall \, \psi \in C_0^\infty(0, +\infty).$$

Finally, we choose $\psi = s\eta$, where $\eta = 1$ in $[0, R]$, $\eta = 0$ in $[2R, \infty)$, $0 \leq \eta \leq 1$ such that

$$|\eta_s| \leq \frac{C}{R} \quad \text{and} \quad |\eta_{ss}| \leq \frac{C}{R^2} \,.$$

Then

$$\int_0^R u^{\frac{2}{3}} \, ds \leq \int_0^\infty u^{\frac{2}{3}} \eta^2 \, ds \leq C \int_R^\infty u^{\frac{2}{3}} \, ds \,,$$

and we conclude that

$$\int_0^{+\infty} u^{\frac{2}{3}} \, ds = +\infty \,.$$

I.e. the metric $\widetilde{g} = u^{\frac{4}{3}} g$ is complete. □

The two previous Lemmas imply that the metric $\widetilde{g} = u^{\frac{4}{3}} g$ is complete and has non-negative 2-Bakry-Emery-Ricci curvature. Hence we obtain a **weighted Bishop-Gromov volume estimate** for a geodesic ball $B_R^{\widetilde{g}}(x_0)$.

Corollary
Let $x_0 \in M^3$. Then, for every $R > 0$, there exists $C > 0$ such that

$$\mathrm{Vol}_f\left(B_R^{\widetilde{g}}(x_0)\right) := \int_{B_R^{\widetilde{g}}(x_0)} e^{-f}\, dV_{\widetilde{g}} \leq CR^5\,,$$

where $f = \frac{2}{3}\log u$. Equivalently, in terms of u and the volume form of g,

$$\int_{B_R^{\widetilde{g}}(x_0)} u^{\frac{4}{3}}\, dV_g \leq CR^5\,.$$

Follows from well-known comparison results e.g. in *Qian (1997)*, *Lott (2003)*, *Bakry-Qian (2005)*, *Wei-Wylie (2009)*.

We have seen that Schoen-Simon-Yau proved the following estimate

$$\int_M |A|^{5+\delta}\phi^{5+\delta}\, dV_g \le C\int_M |\nabla\phi|^{5+\delta}\, dV_g \qquad \forall\phi\in C_0^\infty(M).$$

Lemma
For every $\delta > 0$ small enough, there exists $C > 0$ such that

$$\int_M |A|^{5+\delta} u^{-2-\frac{2\delta}{3}}\phi^{5+\delta}\, dV_g \le C\int_M u^{-2-\frac{2\delta}{3}}|\nabla\phi|^{5+\delta}\, dV_g \quad \forall\,\phi\in C_0^\infty(M).$$

Idea of the proof: from *Schoen-Simon-Yau (1975)* proof

$$\int_M |A|^p\psi^2 \le C\int_M |A|^{p-2}|\nabla\psi|^2 \qquad \forall\,\psi\in C_0^\infty(M),$$

for every $p\in[4, 4+\sqrt{8/n}]$ and for some $C>0$. We test this inequality with $\psi = u^\alpha\phi$, $\phi\in C_0^\infty(M)$, with u the positive solution to

$$-\Delta u = |A|^2 u$$

and $\alpha < 0$. Using this equation and Simons' identity, after *some* estimates, we get the result. □

Proof of the stable Bernstein Theorem:

let $x_0 \in M^3$, and let $\widetilde{r}$ denotes the distance function from x_0 with respect to the metric $\widetilde{g} = u^{\frac{4}{3}} g$. We choose

$$\phi := \eta(\widetilde{r}) \,,$$

where $0 \leq \eta \leq 1$, $\eta = 1$ on $[0, R]$, $\eta = 0$ on $[2R, \infty)$ and $|\eta'| \leq \frac{C}{R}$, for some $C, R > 0$. From the weighted integral estimate, we have

$$\begin{aligned}
\int_M |A|^{5+\delta} u^{-2-\frac{2\delta}{3}} \phi^{5+\delta} \, dV_g &\leq C \int_M u^{-2-\frac{2\delta}{3}} |\nabla \phi|_g^{5+\delta} \, dV_g \\
&= C \int_M u^{-2-\frac{2\delta}{3}+\frac{2(5+\delta)}{3}} |\widetilde{\nabla} \phi|_{\widetilde{g}}^{5+\delta} \, dV_g \\
&\leq \frac{C}{R^{5+\delta}} \int_{B^{\widetilde{g}}_{2R}(x_0)} u^{\frac{4}{3}} \, dV_g \\
&\leq \frac{C}{R^{\delta}} \,,
\end{aligned}$$

(we used that $|\widetilde{\nabla} \widetilde{r}|_{\widetilde{g}} = 1$). Since $\delta > 0$, letting $R \to \infty$, we get

$$|A| \equiv 0 \quad \text{on } M^3,$$

and this concludes the proof. □

- The cases $n = 4, 5, 6$ are **open**.

- The proof by *Chodosh-Li (2021)* is based on the **non-parabolicity of** M^3 (i.e. M^3 admits a positive Green's function G for the Laplacian). They perform careful estimates of the quantity:

$$F(t) := \int_{\Sigma_t} |\nabla G|^2 \,,$$

  where Σ_t is the $t-$level set of G (these are $2-$dimensional). The estimates seem to work only in dimension $n = 3$. They finally test the stability inequality derived by Schoen-Simon-Yau with $\psi = \eta(G)$ to get rigidity.

- We used a **different approach** which seems **suitable to be adapted also to other dimensions.**

- We think this conformal method can be applied for other problems. For instance, very recently we managed to prove the following:

  Theorem [C.-Mastrolia-Monticelli, in preparation]
  Let (M^n, g), $n \geq 10$, be a complete critical metric of the functional

$$\mathfrak{S}^2 = \int R_g^2 dV_g$$

  with finite energy, i.e. $R_g \in L^2(M^n)$. Then (M^n, g) is scalar flat, and thus a global minimum of the functional $\mathfrak{S}^2$.

This conformal deformation can be applied also in the case of minimal immersion $M^n \hookrightarrow (X^{n+1}, h)$ with *finite index*, i.e. the number of negative eigenvalues (counted with multiplicity) of the Jacobi operator

$$\Delta + |A|^2 + \mathrm{Ric}_h(\nu, \nu)\,,$$

on every compact domain in M (with Dirichlet boundary conditions) is finite (ν is a unit normal vector to M in X).

Remark: stability $\Longrightarrow$ finite (zero) index.

Theorem [C.-Mastrolia-Roncoroni (2022)]
If (X^{n+1}, h) is a closed $(n+1)-$dimensional manifold with $n \leq 5$ and such that

$$\mathrm{Sec}_h \geq 0 \quad \textit{and} \quad \mathrm{Ric}_h > 0\,.$$

Then every complete, orientable, immersed, minimal hypersurface $M^n \hookrightarrow (X^{n+1}, h)$ with finite index must be compact.

Corollary
Under the assumptions of the previous theorem, there is no complete, orientable, stable minimal hypersurface $M^n \hookrightarrow (X^{n+1}, h)$.

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In particular, there is no complete, orientable, stable minimal hypersurface of the round spheres $M^n \hookrightarrow (\mathbb{S}^{n+1}, g_{\mathrm{std}})$, provided $n \leq 5$.

- If $n = 2$: known and follows from *Schoen-Yau (1982)*;
- if $n = 3$: known and follows by a recent result by *Chodosh-Li-Striker (2022)*;
- if $n = 4, 5$: new (?!);
- if $n > 5$: open (?!).

Theorem [C.-Mastrolia-Roncoroni (2022)]
If (X^{n+1}, h) is a closed $(n+1)-$dimensional manifold with $n \leq 5$ and such that

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Then every complete, orientable, immersed, minimal hypersurface $M^n \hookrightarrow (X^{n+1}, h)$ with finite index must be compact.

Proof. Suppose, by contradiction, that M is non-compact. Then, there exist $0 < u \in C^\infty(M)$ and $K \subset M$ compact subset such that

$$-\Delta u = \left[|A|^2 + \mathrm{Ric}_h(\nu,\nu)\right] u \quad \text{ in } M \setminus K.$$

Let $k > 0$ and consider the conformal metric

$$\widetilde{g} = u^{2k} g\,,$$

where g is the induced metric on M. As shown in *Fischer-Colbrie (1985)*, one can construct a minimizing geodesic in the metric $\widetilde{g} = u^{2k} g$, $\gamma = \gamma(s) : [0, \infty) \to M \setminus K$, where s is the $g-$arclength, such that it has infinite length in the metric g.

By the second variation formula (cfr. *Elbert-Nelli-Rosenberg (2007)*)

$$(n-1)\int_0^a (\varphi_s)^2\, ds \geq k(n-3)\int_0^a \varphi\varphi_s \frac{u_s}{u}\, ds$$
$$+\frac{k\,[4-k(n-1)]}{4}\int_0^a \varphi^2 \left(\frac{u_s}{u}\right)^2 ds + \int_0^a \varphi^2 \left(kRic_h(\nu,\nu) + \sum_{j=2}^n R^h_{1j1j} \right) ds$$
$$+\int_0^a \varphi^2 \left(k|A|^2 - A_{11}^2 - \sum_{j=2}^n A_{1j}^2 \right) ds,$$

for every smooth function φ such that $\varphi(0) = \varphi(a) = 0$ and for every $k > 0$. Since $\mathrm{Sec}_h \geq 0$ and $\mathrm{Ric}_h(\nu,\nu) \geq R_0 > 0$, we obtain

$$(n-1)\int_0^a (\varphi_s)^2\, ds \geq k(n-3)\int_0^a \varphi\varphi_s \frac{u_s}{u}\, ds$$
$$+\frac{k\,[4-k(n-1)]}{4}\int_0^a \varphi^2 \left(\frac{u_s}{u}\right)^2 ds$$
$$+\int_0^a \varphi^2 \left(kR_0 + k|A|^2 - A_{11}^2 - \sum_{j=2}^n A_{1j}^2 \right) ds.$$

Being A trace-free and choosing $k = \frac{n-1}{n}$, we get

$$\int_0^a (\varphi_s)^2 \, ds \geq \frac{n-3}{n} \int_0^a \varphi \varphi_s \frac{u_s}{u} \, ds + \frac{6n - n^2 - 1}{4n^2} \int_0^a \varphi^2 \left(\frac{u_s}{u} \right)^2 \, ds + \frac{R_0}{n} \int_0^a \varphi^2 \, ds.$$

If $n \leq 5$, we have

$$\frac{6n - n^2 - 1}{4n^2} \geq \delta_0 > 0,$$

moreover, there exists $C > 0$ such that

$$\frac{n-3}{n} \varphi \varphi_s \frac{u_s}{u} \geq -\delta_0 \varphi^2 \left(\frac{u_s}{u} \right)^2 - C(\varphi_s)^2.$$

Therefore, there exists $C > 0$ such that

$$C \int_0^a (\varphi_s)^2 \, ds \geq \frac{R_0}{n} \int_0^a \varphi^2 \, ds$$

for every smooth function φ such that $\varphi(0) = \varphi(a) = 0$.

Integrating by parts we obtain

$$\int_0^a \left(\varphi\varphi_s s + CR_0\varphi^2\right) ds \leq 0.$$

Choosing $\varphi(s) = \sin(\pi s\, a^{-1})$, $s \in [0, a]$ one has

$$\left(CR_0 - \frac{\pi^2}{a^2}\right) \int_0^a \sin^2(\pi s\, a^{-1}) ds \leq 0$$

i.e.

$$a^2 \leq \frac{\pi^2}{CR_0}.$$

We conclude that the length (in the metric g) of the geodesic $\widetilde{\gamma}(s)$ is finite and this gives a contradiction. Therefore (M^n, g) must be compact and this concludes the proof.

$\square$

Corollary
Under the assumptions of the previous theorem, there is no complete, orientable, stable minimal hypersurface $M^n \hookrightarrow (X^{n+1}, h)$.

Proof. If M is stable, by the previous Theorem it must be compact. Moreover, there exists $0 < u \in C^\infty(M)$ such that

$$-\Delta u = \left[|A|^2 + \mathrm{Ric}_h(\nu, \nu)\right] u \quad \text{in } M.$$

Integrating over M we get a contradiction, since $\mathrm{Ric}_h > 0$. Equivalently, one can use $f \equiv 1$ in the stability inequality to get a contradiction.

□